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Bayesian Estimation for Zero-Truncated Bivariate Poisson Regression Model

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Abstract

Bivariate count data occurs when two associated variable counts necessitate joint estimate primarily for efficiency purposes. This paper presents Bayesian estimate for the zero-truncated bivariate Poisson regression model. This bivariate model was established using marginal-conditional models. Bayes estimators were executed utilizing the random walk Metropolis-Hastings algorithm with two distinct prior distributions: Laplace and normal distributions. Moreover, estimators employing the bootstrap approach were proposed. Additionally, the credible intervals and the percentile bootstrap confidence intervals were analyzed. The performance of the Bayes estimators was compared with that of the bootstrap estimators and the maximum likelihood estimators via a Monte Carlo simulation analysis, focusing on mean square error. The performance of intervals was evaluated based on coverage probability and average length. Furthermore, the explanatory variables were produced under conditions of both multicollinearity and a lack of multicollinearity. Two empirical datasets were examined to demonstrate the practical use of the suggested model and methodology. The findings from both the simulation and application indicate that the Bayesian method with a normal prior distribution surpasses alternative methods.

Keywords:

Bayesian Estimation;
Bootstrap Method;
Count Data;
Maximum Likelihood Estimation;
Metropolis-Hastings Algorithm.

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1- Introduction

The interest in discrete bivariate count models has been greatly increased in recent years. Bivariate count models are employed in many real-life situations where paired counts exhibit correlation and necessitate joint estimates. For instance, there are statistics regarding the number of vehicles involved in road accidents and the number of casualties. In addition, the number of children in families with at least one child attending school and the number of schools attended by children in the family. Recently, Lerdsuwansri et al. (2022) [1] proposed a Conway-Maxwell-Poisson regression model for road traffic injuries in Thailand, whereas Simmachan et al. (2022) [2] suggested modeling road accident fatalities with underdispersion and zero-inflated counts. The widely used distribution for modeling the bivariate count data sets is the Poisson distribution. For example, Jung & Winkelmann (1993) [3] demonstrated that bivariate Poisson regression is superior to the univariate specification. Ho & Singer (1997) [4] proposed bivariate Poisson and bivariate Poisson log-normal regression models for the analysis of counts derived from a stratified sampling scheme. Kocherlakota & Kocherlakota (2001) [5] developed regression coefficient estimators for a bivariate Poisson distribution under various conditions, including unrestricted linear models, parallelism of regression planes, and coincidence of regression planes. Famoye (2012) [6] proposed a novel bivariate generalized Poisson regression model applicable to both over-dispersed and under-dispersed data.

AlMuhayfith et al. (2016) [7] examined parameter estimation for bivariate and zero-inflated bivariate Poisson regression models through the conditional method. Qarmalah & Alzaid (2023) [8] proposed a class of bivariate Poisson models derived from the bivariate Bernoulli model, capable of modeling both positively and negatively correlated data. Additionally, a bivariate model was constructed using the marginal-conditional models [9]. Islam & Chowdhury (2015)

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[10] recently proposed the marginal-conditional model for bivariate Poisson distribution. It is well known that zero-inflated distribution can be applied to the models of data sets that contain lots of zeros, while zero-truncated distribution is proper for data sets that contain no zeros. Chowdhury & Islam (2016) [11] proposed a zero-truncated bivariate Poisson model for count data, which was performed using marginal-conditional models on the bivariate model. Moreover, a covariate-dependent bivariate generalized linear model is applied to road safety data utilizing canonical link functions via maximum likelihood estimation of parameters for Poisson distribution, but the other estimates in this model have yet to be discovered.

The maximum likelihood estimation of parameters is valid for an asymptotically large sample size of data [12, 13]. A frequent issue in count regression models is that maximum likelihood estimates can become unstable and exhibit large standard errors of the estimates, which negatively impact statistical inference when sample sizes are inadequate. Moreover, the presence of multicollinearity among the explanatory variables in regression modeling has undesirable effects on maximum likelihood estimators which reduce the reliability of statistical inferences. Hybrid of the multicollinearity and high-dimensional data problems can lead to instabilities in a predictive model when applied to a new data set [14, 15]. To overcome the problem, various alternatives to parameter estimation have been proposed, and the bootstrap method is one of them. The bootstrap method is the technique most widely applied to remedy the effect of the small sample sizes and multicollinearity [16, 17]. In a ridge regression model, robust methods are combined with bootstrapping to address the simultaneous issues of multicollinearity and multiples outliers in the data [18]. The shrinkage parameter estimators were proposed by Sudjai & Duangsaphon (2020) [19], which also used the bootstrapping method to estimate the Liu logistic regression coefficient with multicollinearity problem. Moreover, Perveen & Suhail (2021) [20] proposed some new Poisson bootstrap Liu and Ridge estimators to address the problem of multicollinearity in the Poisson regression model.

The zero-truncated bivariate Poisson model is well-suited for discrete, nonzero integer-valued bivariate count data. However, existing literature has largely overlooked Bayesian inference in this context. Among the contributions to Bayesian estimation of bivariate regression models, Majumdar & Gries (2010) [21] proposed simulation to explore the effectiveness of parameter estimation using the Bayesian procedure for bivariate zero-inflated Poisson regression, comparing the performance to the Bayesian and classical approaches. Choe et al. (2012) [22] estimated the regression coefficients of a bivariate Poisson regression model utilizing a Bayesian procedure that employs the Metropolis-Hastings algorithm within the Gibbs sampler to derive samples from the full conditional distributions. Specifically, these priors exhibit characteristics of high spread, exemplified by a normal density function with an exceptionally large variance. The posterior means and the highest posterior density (HPD) intervals for each parameter were computed; however, the performance and sensitivity assessments, including mean square error, coverage probability, and average length, remain to be determined. Arnold & Ghosh (2023) [23] proposed the Bayesian and frequentist approaches for bivariate Poisson conditional distributions. Additionally, recommendations regarding the selection of hyperparameters for the prior distribution were provided. Undiscovered Bayesian estimates in a zero-truncated bivariate Poisson regression model. Furthermore, several studies illustrated the application of a Bayesian methodology in the count regression model. Supharakonsakun (2021) [24] suggested the Bayesian estimation for Poisson distribution.

Thangjai et al. (2021) [25] introduced confidence intervals for coefficients of variation of PM10 dispersion. Recently, Chaiprasithikul & Duangsaphon (2022) [26] proposed the Metropolis-Hastings algorithm for estimating Bayes estimators in the discrete Weibull regression model with censored data, utilizing two different prior distributions: uniform noninformative and normal distributions. The results of this model demonstrate that a normal prior distribution provides a better fit than a uniform noninformative prior distribution. Chaiprasithikul & Duangsaphon (2022) [27] proposed zero-inflated and hurdle discrete Weibull regression models utilizing the Metropolis-Hastings algorithm with three distinct prior distributions: uniform noninformative, Laplace, and normal distributions. Duangsaphon et al. (2024) [28] similarly presented the median discrete Weibull regression model. The findings of this model indicate that the Laplace prior distribution provides a superior fit compared to other prior distributions. Moreover, Srisuradetchai & Niyomdecha (2024) [29] proposed Bayesian inference for the gamma zero-truncated Poisson distribution.

In the present article, the Bayesian estimation for zero-truncated bivariate Poisson model based on different prior distributions is proposed. What's more, the bootstrap method is examined. To investigate the performance of Bayesian estimation under various prior distributions, the bootstrap method, and maximum likelihood estimation, a simulation study was carried out. In addition, two real data sets were analyzed to see how the model works in practice.

The remainder of this paper is organized as follows. Section 2 provides an overview of the zero-truncated bivariate Poisson regression model. The maximum likelihood estimation is shown in section 3. The bootstrap approach is suggested in section 4. Bayesian estimate is demonstrated in section 5. Section 6 reports a simulation study to examine the performance of various estimations in both multicollinearity and no multicollinearity scenarios. Additionally, Section 7 applies this model to two real data sets. Lastly, Sections 8–9 provide the discussion and concluding remarks, respectively.

2- Zero-Truncated Bivariate Poisson Regression Model

This section provides the basic framework for zero-truncated bivariate Poisson regression model see Chowdhury and Islam (2016) [11].

2-1- Bivariate Poisson Model

Consider two jointly distributed random variables, Y_1 and Y_2, each denotes event counts. Let Y_1 be the number of occurrences of the first event in a given interval following a Poisson distribution with parameter λ_1. The probability mass function of Y_1 is:

$$g_1(y_1) = \frac{e^{-\lambda_1}{\lambda_1}^{y_1}}{y_1!};\ y_1 = 0,1,\ldots,\ \lambda_1 > 0. \tag{1}$$

Let Y_{2i} be a random variable associated with the number of occurrences of the second event in a given interval resulting from the $i-th$ occurrence of the first event, and suppose that Y_{2i} has a Poisson distribution with parameterλ_2 that can be defined as follows:

$$g_2(y_{2i}) = \frac{e^{-\lambda_2}\lambda_2^{y_{2i}}}{y_{2i}!};\ y_{2i} = 0,1,\ldots,\ \lambda_2 > 0. \tag{2}$$

Now, if Y_{2i} are assumed to be mutually independent, then the conditional distribution of $Y_2 = Y_{21} + Y_{22} + \ldots + Y_{2y_1}$, the total number of occurrences of the second event recorded among the Y_1 event occurring in the $j-th$time interval, is Poisson distribution with parameter$\lambda_2 y_1$. Thus, the conditional distribution of Y_2given Y_1 are as follows:

$$g(y_2|y_1) = \frac{e^{-\lambda_2 y_1}(\lambda_2 y_1)^{y_2}}{y_2!};\ y_2 = 0,1,\ldots. \tag{3}$$

Using conditional and marginal model, the joint distribution of the number of occurrences of the first event Y_1 and the corresponding number of occurrences of the second event Y_2 can be expressed as follows:

$$g(y_1, y_2) = g(y_2|y_1)\cdot g_1(y_1) = \frac{e^{-\lambda_1}{\lambda_1}^{y_1}e^{-\lambda_2 y_1}(\lambda_2 y_1)^{y_2}}{y_1!y_2!} \tag{4}$$

where $y_1 = 0,1,\ldots$ and $y_2 = 0,1,\ldots$.

2-2- Zero Truncated Bivariate Poisson (ZTBVP) Model

The probability of $Y_1 = 0$ is $e^{-\lambda_1}$, using Equation 1. Given Y_{ZT1} is a zero-truncated Poisson random variable based on $Y_1$and Y_{ZT2} is a zero-truncated Poisson random variable based on Y_2. Then, the probability mass function for Y_{ZT1} is as follows:

$$g_1^*(y_1) = P(Y_1 = y_1|Y_1 > 0) = \frac{P(Y_1=y_1)}{P(Y_1>0)} = \frac{P(Y_1=y_1)}{1-P(Y_1=0)} = \frac{{\lambda_1}^{y_1}}{y_1!(e^{\lambda_1}-1)}. \tag{5}$$

Then, the exponential form of the probability mass function of Y_{ZT1} can be expressed as follows:

$$g_1^*(y_1) = exp\left[y_1\, ln\,\lambda_1 - ln(y_1!) - ln\left(e^{\lambda_1} - 1\right)\right]. \tag{6}$$

The mean and variance of Y_{ZT1} can be shown as follow:

$$\mu_{Y_{ZT1}} = E[Y_{ZT1}] = \frac{\lambda_1 e^{\lambda_1}}{(e^{\lambda_1}-1)} \tag{7}$$

and

$$\sigma^2_{Y_{ZT1}} = Var[Y_{ZT1}] = \frac{\lambda_1 e^{\lambda_1}}{(e^{\lambda_1}-1)}\left(1 - \frac{\lambda_1}{e^{\lambda_1}-1}\right). \tag{8}$$

In Equation 3, the zero-truncated conditional model of $Y_2 = y_2|y_1, Y_2 > 0$ can be rewritten as follows:

$$P(Y_2 = y_2|y_1, Y_2 > 0) = \frac{P(Y_2=y_2|Y_1=y_1)}{P(Y_2>0|Y_1=y_1)} = \frac{P(Y_2=y_2|Y_1=y_1)}{1-P(Y_2=0|Y_1=y_1)}. \tag{9}$$

Then, the probability mass function for zero-truncated conditional Poisson distribution $(Y_{ZT2}|Y_{ZT1})$ can be shown as follows:

$$g_2^*(y_2|y_1) = \frac{e^{-\lambda_2 y_1}(\lambda_2 y_1)^{y_2}}{y_2!} \times \frac{1}{(1-e^{-\lambda_2 y_1})} = \frac{(\lambda_2 y_1)^{y_2}}{y_2!(e^{\lambda_2 y_1}-1)}. \tag{10}$$

where $y_1 = 1,2,\ldots$ and $y_2 = 1,2,\ldots$.

The mean and variance of $Y_{ZT2}|Y_{ZT1}$ are

$$\mu_{Y_{ZT2}|Y_{ZT1}} = E[Y_{ZT2}|Y_{ZT1}] = \frac{\lambda_2 y_1 e^{\lambda_2 y_1}}{(e^{\lambda_2 y_1}-1)} \tag{11}$$

and

$$\sigma^2_{Y_{ZT2}|Y_{ZT1}} = Var[Y_{ZT2}|Y_{ZT1}] = \frac{\lambda_2 y_1 e^{\lambda_2 y_1}}{(e^{\lambda_2 y_1}-1)}\left(1-\frac{\lambda_2 y_1}{e^{\lambda_2 y_1}-1}\right). \quad (12)$$

In a comparable manner, the joint distribution construction of employing the zero-truncation distribution's marginal and conditional distribution for bivariate Poisson model can be obtained as follows:

$$g^*(y_1, y_2) = g_2^*(y_2|y_1)g_1^*(y_1) = \frac{(\lambda_2 y_1)^{y_2}(\lambda_1)^{y_1}}{y_1!y_2!(e^{\lambda_2 y_1}-1)(e^{\lambda_1}-1)} \quad (13)$$

where $y_1 = 1,2,\ldots$ and $y_2 = 1,2,\ldots$.

The zero truncated bivariate Poisson model expression in Equation 13 can be expressed in bivariate exponential form as:

$$g^*(y_1, y_2) = exp[\,y_1 \ln\lambda_1 - \ln(y_1!) - \ln(e^{\lambda_1} - 1) + y_2 \ln\lambda_2 + y_2 \ln(y_1) - \ln(y_2!) - \ln(e^{\lambda_2 y_1} - 1)]. \quad (14)$$

2-3- Regression Model

Given the k explanatory variables, $\boldsymbol{x}' = (1, x_1, \ldots, x_k)$ and a vector composed of regression coefficients is $\beta_1{}' = (\beta_{10}, \beta_{11}, \ldots, \beta_{1k})$ and $\beta_2{}' = (\beta_{20}, \beta_{21}, \ldots, \beta_{2k})$, it is assumed that the parameter λ_1 and λ_2 are related to k explanatory variables $\boldsymbol{x}'$as follows:

$$\ln\lambda_1 = \boldsymbol{x}'\beta_1 \text{ and } \ln\lambda_2 = \boldsymbol{x}'\beta_2. \quad (15)$$

From Equation 14 and 15, the joint probability mass function of (Y_{ZT1}, Y_{ZT2}) given $\boldsymbol{x}$ can be written as follows:

$$g^*(y_1, y_2|\boldsymbol{x}) = exp\begin{bmatrix} y_1 \ln\left(e^{(x'\boldsymbol{\beta}_1)}\right) - \ln(y_1!) - \ln(e^{e^{(x'\boldsymbol{\beta}_1)}} - 1) \\ +y_2 \ln\left(e^{(x'\boldsymbol{\beta}_2)}\right) + y_2 \ln(y_1) - \ln(y_2!) - \ln(e^{y_1 e^{(x'\boldsymbol{\beta}_2)}} - 1) \end{bmatrix}. \quad (16)$$

3- Maximum Likelihood Estimation

In this section, the maximum likelihood estimation for the ZTBVP regression model is performed.

Given a random sample (Y_{ZT1i}, Y_{ZT2i}); $i = 1,2,\ldots,n$ from the ZTBVP distribution with the observed values $\boldsymbol{y_1} = y_{11}, y_{12}, \ldots, y_{1n}$, $\boldsymbol{y_2} = y_{21}, y_{22}, \ldots, y_{2n}$, the explanatory variables $\boldsymbol{x}_i' = (1, x_{i1}, \ldots, x_{ik})$; $i = 1,2,\ldots,n$, and $\boldsymbol{\theta} = (\beta_{10}, \ldots, \beta_{1k}, \beta_{20}, \ldots, \beta_{2k})$. From Equation 16, the likelihood function of the zero truncated bivariate Poisson regression model is given by

$$L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x}) = \prod_{i=1}^n exp\begin{bmatrix} y_{1i} \ln\left(e^{x_i'\boldsymbol{\beta_1}}\right) - \ln(y_{1i}!) - \ln(e^{e^{x_i'\boldsymbol{\beta_1}}} - 1) \\ +y_{2i} \ln\left(e^{x_i'\boldsymbol{\beta_2}}\right) + y_{2i} \ln(y_{1i}) - \ln(y_{2i}!) - \ln(e^{y_{1i}e^{x_i'\boldsymbol{\beta_2}}} - 1) \end{bmatrix} \quad (17)$$

The log-likelihood function of the zero-truncated bivariate Poisson regression model is given by

$$\ln L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x}) = \sum_{i=1}^n \begin{bmatrix} y_{1i} \ln(\boldsymbol{x}_i'\boldsymbol{\beta_1}) - \ln(y_{1i}!) - \ln(e^{e^{x_i'\boldsymbol{\beta_1}}} - 1) \\ +y_{2i} \ln(\boldsymbol{x}_i'\boldsymbol{\beta_2}) + y_{2i} \ln(y_{1i}) - \ln(y_{2i}!) - \ln(e^{y_{1i}e^{x_i'\boldsymbol{\beta_2}}} - 1) \end{bmatrix}. \quad (18)$$

The maximum likelihood estimators are obtained by setting the first partial derivatives of the log-likelihood function with respect to each unknown parameter equal to zero. The first partial derivatives of $\ln L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})$ with respect to parameters β_{1j} and $\beta_{2j}, j = 0,1,..,k$, are showed as follows:

$$\frac{\partial}{\partial\beta_{1j}} \ln L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x}) = \sum_{i=1}^n \left[y_{1i} - \frac{e^{x_{ij}'\beta_1} e^{e^{x_{ij}'\beta_1}}}{e^{e^{x_{ij}'\beta_1}} - 1} \right] x_{ij} = 0 \quad (19)$$

and

$$\frac{\partial}{\partial\beta_{2j}} \ln L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x}) = \sum_{i=1}^n \left[y_{2i} - \frac{y_{1i}e^{x_{ij}'\beta_2} e^{y_{1i}e^{x_{ij}'\beta_2}}}{e^{y_{1i}e^{x_{ij}'\beta_2}} - 1} \right] x_{ij} = 0. \quad (20)$$

It is apparent there is no explicit solution to the aforementioned equations. Numerical methods such as the Newton-Raphson or Gauss-Newton methods, or their variations, can be used to solve equations.

We define the observed Fisher's information matrix at $\boldsymbol{\theta}$as $I(\boldsymbol{\theta})$ which contain the negative of the second derivative of the log-likelihood function. Consequently, the variance-covariance matrix of maximum likelihood estimators is the inverse of the observed Fisher's information matrix,

$$\Sigma = I^{-1}(\boldsymbol{\theta}). \tag{21}$$

The maximum likelihood estimators are replaced, yielding an estimator of Σ presented by $\hat{\Sigma}$. Parameter inferences are conducted utilizing the maximum likelihood method. Under certain regularity conditions, these estimators exhibit standard asymptotic properties [30]. Thus, by the asymptotic normality of maximum likelihood estimators, the $100(1-\alpha)\%$ confidence intervals for parameters β_{ij} respectively manifests as;

$$\hat{\beta}_{ij} \pm z_{\alpha/2}\sqrt{\hat{\sigma}^2_{\beta_{ij}}} \tag{22}$$

where $z_{\alpha/2}$ is the upper $\alpha/2-th$ quantile of the standard normal distribution, $i=1,2, j=0,1,..,k$ and $\hat{\sigma}_{\beta_{ij}}$is standard error of estimator $\hat{\beta}_{ij}$which can be obtained from $\hat{\Sigma}$.

4- Bootstrapping Method

Given a random sample (Y_{ZT1i}, Y_{ZT2i}); $i=1,2,\dots,n$ from the ZTBVP distribution with the observed values $\boldsymbol{y_1} = y_{11}, y_{12},\dots,y_{1n}, \boldsymbol{y_2} = y_{21}, y_{22},\dots,y_{2n}$, the explanatory variables $\boldsymbol{x}'_i = (1, x_{i1},\dots,x_{ik})$; $i=1,2,\dots,n$, and β_{ij} is a regression coefficient; $i=1,2, j=0,1,\dots,k$. There are two approaches for bootstrapping: the first approach involves resampling the random error term, while the second approach resamples from the observations. This study adopts the second approach. Consequently, the outlined procedure for bootstrapping is as follows:

Step 1: Create a bootstrap sample of size n $(z_1^*, z_2^*,\dots,z_n^*)$ from the original data with the replacement giving $\frac{1}{n}$ probability for each z_i^*. Thus, this study obtains the following: $z_i^* = (\boldsymbol{y_1}_i^*, \boldsymbol{y_2}_i^*, \boldsymbol{x}_i'^*)$, $i=1,2,\dots,n$.

Step 2: Estimate parameters β_{1j} and β_{2j}, $j=0,1,\dots,k$ for the zero-truncated bivariate Poisson regression model using maximum likelihood method.

Step 3: Repeat steps 1-2 for B times, where B is the number of repetitions. This investigation can therefore arise bootstrap estimates for parameters β_{1j} and β_{2j}.

Step 4: Use the resulting bootstrap estimates in step 3 (e.g. ,$\hat{\beta}_{ij}^{*(1)}, \hat{\beta}_{ij}^{*(2)},\dots,\hat{\beta}_{ij}^{*(B)}$; $i=1,2$, $j=0,1,\dots,k$) to compute the average estimate for each estimator. Therefore, the estimated values of the parameters for application with the bootstrapping method are as follows:

$$\bar{\hat{\beta}}_{ij}^* = \frac{1}{B}\sum_{b=1}^{B}\hat{\beta}_{ij}^{*(b)} \tag{23}$$

The $100(1-\alpha)\%$ percentile bootstrap confidence interval is constructed as follows:

$$\left(\hat{\beta}_{ij}^{*\,\alpha/2}, \hat{\beta}_{ij}^{*\,1-\alpha/2}\right) \tag{24}$$

where $\hat{\beta}_{ij}^{*\,\alpha/2}$ and $\hat{\beta}_{ij}^{*\,1-\alpha/2}$ are $(\alpha/2)B-th$ and $(1-\alpha/2)B-th$ values in the ordered list of the B replications of $\hat{\beta}_{ij}^*$, $i=1,2$, $j=0,1,\dots,k$, and α is the level of significance.

5- Bayesian Estimation

The Bayes estimators for the zero truncated bivariate Poisson regression model are carried out in this section using two schemes of informative prior distributions: Laplace prior distribution and normal prior distribution.

i) Laplace distribution.

This study can execute the informative prior distribution, which should contain all potential values of the β_{ij} parameter, if prior information is available. It chooses the prior distribution of β_{ij}, which is a Laplace distribution with the hyperparameters set to $(0,1/\lambda)$. The possible values of are real numbers that match the possible values of a Laplace distribution. The following the prior distributions are as follows:

$$\pi(\beta_{ij}) = \frac{\lambda}{2}e^{-\lambda|\beta_{ij}|}, \lambda > 0,\ i=1,2, j=0,1,\dots,k.$$

ii) Normal distribution.

As stated earlier, the prior distribution of β_{ij}, which is a normal distribution with the hyperparameters as $\left(\mu_{\beta_{ij}}, \sigma^2_{\beta_{ij}}\right)$, $i=1,2, j=0,1,\dots,k$ is chosen for this study because the possible values of β_{ij} are real numbers that are comparable to the possible values of a normal distribution. The following the prior distributions are as follows:

$$\pi(\beta_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2_{\beta_{ij}}}}e^{-\frac{1}{2\sigma^2_{\beta_{ij}}}\left(\beta_{ij}-\mu_{\beta_{ij}}\right)^2}, \mu_{\beta_{ij}} \in \mathbb{R},\ \sigma^2_{\beta_{ij}} > 0,\ i=1,2, j=0,1,\dots,k.$$

The joint prior distributions of the parameters $\boldsymbol{\theta} = (\beta_{10}, \ldots, \beta_{1k}, \beta_{20}, \ldots, \beta_{2k})$ under the independence assumption is

$$\pi(\boldsymbol{\theta}) = \pi(\beta_{10})\pi(\beta_{11}) \cdots \pi(\beta_{2(k-1)})\pi(\beta_{2k}). \tag{25}$$

The joint posterior density function of the parameters $\boldsymbol{\theta}$ can be written as:

$$p(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x}) = \frac{L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta})}{\iint \cdots \int L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta}) d\beta_{10} \cdots d\beta_{2(k-1)} d\beta_{2k}} \propto L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta}) \tag{26}$$

where $L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})$ is the likelihood function of zero-truncated bivariate Poisson regression model in Equation 17.

The Bayes estimator of each parameter under squared error loss function is the expected value of each parameter under the joint posterior density function. Therefore, the Bayes estimators are given by

$$\hat{\beta}_{ij} = \iint \cdots \int \beta_{ij} p(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x}) d\beta_{10} \cdots d\beta_{2(k-1)} d\beta_{2k} \tag{27}$$

where $i = 1,2, j = 0,1,2, \ldots, k$.

One challenge in executing the Bayesian procedure lies in acquiring the posterior distribution. The process often requires integration, which can be quite challenging to compute, particularly when addressing complex and high-dimensional models. In this context, Metropolis-Hastings (MH) algorithms prove to be extremely useful for modeling deviations from the posterior density and producing precise approximations. [31, 32].

Given that the integral in Equation 26 lacks a closed form, this study chose for the random walk Metropolis-Hastings algorithm to estimate the Bayes estimators. It also determines the joint posterior density function of the parameters $\boldsymbol{\theta} = (\beta_{10}, \ldots, \beta_{1k}, \beta_{20}, \ldots, \beta_{2k})$, $p(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})$ in Equation 26 as the target distribution, while $\boldsymbol{\theta}$ is the current state value, and $\boldsymbol{\theta}^*$ is the proposal value generated from the proposal distribution $q(\boldsymbol{\theta}^*|\boldsymbol{\theta})$. Then, the proposal value $\boldsymbol{\theta}^*$ is accepted with the probability $p = min(1, R_{\boldsymbol{\theta}})$, where

$$R_{\boldsymbol{\theta}} = \frac{L(\boldsymbol{\theta}^*|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta}^*)}{L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta})} \times \frac{q(\boldsymbol{\theta}|\boldsymbol{\theta}^*)}{q(\boldsymbol{\theta}^*|\boldsymbol{\theta})}. \tag{28}$$

For the random walk Metropolis algorithm, the proposal distribution is symmetrical, depending only on the distance between the current state value and the proposal value. Then, the proposal value $\boldsymbol{\theta}^*$ is accepted with probability $p = min(1, R_{\boldsymbol{\theta}})$, where

$$R_{\boldsymbol{\theta}} = \frac{L(\boldsymbol{\theta}^*|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta}^*)}{L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta})}. \tag{29}$$

The iterative steps of the random walk Metropolis algorithm can be described as follows:

Step 1: Establish the initial parameters $\boldsymbol{\theta}^{(0)} = ({\beta_{10}}^{(0)}, \ldots, {\beta_{1k}}^{(0)}, {\beta_{20}}^{(0)}, \ldots, {\beta_{2k}}^{(0)})$ for the algorithm by utilizing the maximum likelihood estimators.

Step 2: For $l = 1,2, \ldots, L$, repeat the following steps;

a) Generate random error vector $\boldsymbol{\varepsilon}$ from a multivariate normal distribution with a zero-mean vector and variance-covariance matrix as a diagonal matrix in which the diagonal elements are the diagonal of the inverse of the observed Fisher's information matrix; $\boldsymbol{\varepsilon} \sim \mathcal{N}\left(\boldsymbol{\mu} = \boldsymbol{0}, \boldsymbol{\Sigma} = diag\left(I^{-1}(\boldsymbol{\theta})\right)\right)$. Then, set $\boldsymbol{\theta}^* = \boldsymbol{\theta}^{(l-1)} + \boldsymbol{\varepsilon}$.

b) Calculate $p = min(1, R_{\boldsymbol{\theta}})$ where $R_{\boldsymbol{\theta}} = \frac{L(\boldsymbol{\theta}^*|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta}^*)}{L(\boldsymbol{\theta}|\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{x})\pi(\boldsymbol{\theta})}$.

c) Generate u from a uniform distribution; $u \sim U(0,1)$.

If $u \leq p$, accept $\boldsymbol{\theta}^*$ and set $\boldsymbol{\theta}^{(l)} = \boldsymbol{\theta}^*$ with probability p.

If $u > p$, reject $\boldsymbol{\theta}^*$ and set $\boldsymbol{\theta}^{(l)} = \boldsymbol{\theta}^{(l-1)}$ with probability $1 - p$.

Step 3: Remove L_0 of the chain for *burn-in*.

Step 4: Calculate the estimated values of the Bayes estimators of the parameters β_{ij} from the average of the generated values as given by;

$$\hat{\beta}_{ij} = \frac{1}{L - L_0} \sum_{l=L_0+1}^{L} {\beta_{ij}}^{(l)} \tag{30}$$

where $i = 1,2, j = 0,1,2, \ldots, k$, L is the number of iterations in the random walk Metropolis algorithm, and L_0 is the burn-in period.

The construction of the Bayesian credible intervals [33, 34] of the parameters β_{ij}, $i = 1,2, , j = 0,1,2, \ldots, k$follows the Monte Carlo procedure. Given an MCMC sample ${\beta_{ij}}^{(l)},\ l = L_0 + 1, L_0 + 2, \dots, L$, the Bayesian credible intervals for β_{ij} can be shown as follows:

Step 1: Sort $\{{\beta_{ij}}^{(l)},\ l = L_0 + 1, L_0 + 2, \dots, L\}$ to obtain the ordered value ${\beta_{ij}}_{(1)} \leq {\beta_{ij}}_{(2)} \leq \dots \leq {\beta_{ij}}_{(L-L_0)}$.

Step 2: Compute the $100(1-\alpha)\%$ credible intervals

$$\left(\hat{\beta}_{ij((L-L_0)\frac{\alpha}{2})}, \hat{\beta}_{ij((L-L_0)(1-\frac{\alpha}{2}))}\right) \tag{31}$$

where $\hat{\beta}_{ij((L-L_0)\frac{\alpha}{2})}$ and $\hat{\beta}_{ij((L-L_0)(1-\frac{\alpha}{2}))}$ are $(L-L_0)\frac{\alpha}{2} - th$ and $(L-L_0)(1-\frac{\alpha}{2}) - th$ values in the ordered list of the $L - L_0$ replications from step 1, $i = 1,2$, $j = 0,1,\dots,k$, and α is the level of significance.

6- Simulation Study

In this section, the Monte Carlo simulation is conducted to assess and compare the performance of estimating the parameters of the zero- truncated bivariate Poisson regression model; maximum likelihood method, bootstrapping method, and the Bayesian methods. The simulation was executed, based on the conditions as follows:

(a) The various selected sample sizes (n) are 50, 100, and 200 for two explanatory variables.

(b) The explanatory variables are considered, based on both no multicollinearity (Scenario 1) and multicollinearity (Scenario 2);

Scenario 1: Set two different explanatory variables from a normal distribution$X_1 \sim N(0,1)$, and a uniform distribution $X_2 \sim U(0,10)$. Set the regression parameters to take values $\boldsymbol{\theta} = (\beta_{10}, \beta_{11}, \beta_{12}, \beta_{20}, \beta_{21}, \beta_{22})$ = (0.2,-0.1,0.1,-0.3,0.2,0.1).

Scenario 2: Set two different explanatory variables from a multivariate uniform distribution $\boldsymbol{X} \sim U(0,1)$ where $\boldsymbol{X} = (X_1, X_2)$ and varying correlation level as 0.3 0.5 0.7 and 0.95. Setting the regression parameters to take values $\boldsymbol{\theta} = (\beta_{10}, \beta_{11}, \beta_{12}, \beta_{20}, \beta_{21}, \beta_{22}) = (1.2,-0.45,0.2,-0.63,0.21,0.12)$.

(c) Generate two count response variables from the zero-truncated bivariate Poisson model using package "actuar" in R to derive response variables from marginal and conditional distributions as in Equation 6 and 10. This study assumed that the parameter λ_1 and λ_2 are related to k explanatory variables $\boldsymbol{x}'$ as in Equation 15; so, it used $\lambda_{1i} = e^{\boldsymbol{x}'_i \boldsymbol{\beta}_1}, \lambda_{2i} = e^{\boldsymbol{x}'_i \boldsymbol{\beta}_2}$; $i = 1,2,\dots,n$.

(d) The parameters are estimated by using the numerical method for maximum likelihood estimation. This study calculates the maximum likelihood estimators by minimizing the negative log-likelihood function of the zero truncated bivariate Poisson regression model using function $optim()$from package "stats" in R; it presents the BFGS method. Moreover, this study calculates $\hat{\Sigma}$ by inverting the Hessian matrix from the function, i.e. the Hessian matrix is the observed Fisher's information matrix.

(e) For the Bayesian estimation, it fixes the hyperparameters' values of β_{ij}.

The choice of the hyperparameters' values is generally modified by available information of dataset to improve the Bayes estimators. It fixes the hyperparameters' values of β_{ij},$i = 1,2, j = 0,1,2,\dots,k$ for normal prior distribution with mean zero, and variance is one. For Laplace prior distribution, it fixes the hyperparameters' values of β_{ij} $i = 1,2, j = 0,1,2,\dots,k$ with λ is 0.5.

(f) $100(1-\alpha)\%$ confidence interval is 95%

(g) This study considers $L = 10{,}000$ iterations of the sampler and uses the first 10% of the data as burn-in,

$L_0 = 1{,}000$.

(h) This simulation study is repeated 1,000 times.

The measures of accuracy [24, 25] for the estimators are

(i) the estimates of the parameters (Est.) $= \frac{\sum_{l=1}^{1,000} {\hat{\beta}_{ij}}^{(l)}}{1,000}$, (32)

(ii) the mean square error (MSE) $= \frac{\sum_{l=1}^{1,000} ({\hat{\beta}_{ij}}^{(l)} - \beta_{ij})^2}{1,000}$, (33)

(iii) the coverage probability (CP) $= \frac{\#\{LCL_{\beta_{ij}} < \beta_{ij} < UCL_{\beta_{ij}}\}}{1,000}$, (34)

and

(iv) the average length (AL) $= \frac{\sum_{l=1}^{1,000} ({UCL_{\beta_{ij}}}^{(l)} - {LCL_{\beta_{ij}}}^{(l)})}{1,000}$, (35) where $\hat{\beta}_{ij}^{(l)}$ is the j-th estimator, $LCL_{\beta_{ij}}^{(l)}$ and $UCL_{\beta_{ij}}^{(l)}$ are the j-th lower bound and upper bound for the 95% confidence interval of the l-th time, and $\#\{LCL_{\beta_{ij}} < \beta_{ij} < UCL_{\beta_{ij}}\}$ is the total of the number of times that the true value β_{ij} falls within the confidence interval bounds. The same measure of accuracy is utilized for the estimators of regression parameters across each approach, encompassing maximum likelihood estimation (MLE), Bootstrapping (Bootstrap), and Bayesian methods, while considering various prior distributions: Laplace prior (Bayes(L)) and normal prior (Bayes(N)). A comparison of their performances is presented in Figure 1. Tables 1 to 5 provide the estimates of the parameters (Est.) and the mean squared error (MSE), while Tables 6 to 10 display the 95% coverage probability (CP) and the average length (AL). The accuracy measures for the estimators include: the parameter estimates (Est.) being in proximity to the true parameter values. A larger sample size leads to lower estimated MSE values. Additionally, the minimum MSE value. The CP typically aligns with the nominal confidence level (95%). With larger sample sizes, the AL of the 95% confidence intervals diminishes. Furthermore, the shortest AL value is also noted.

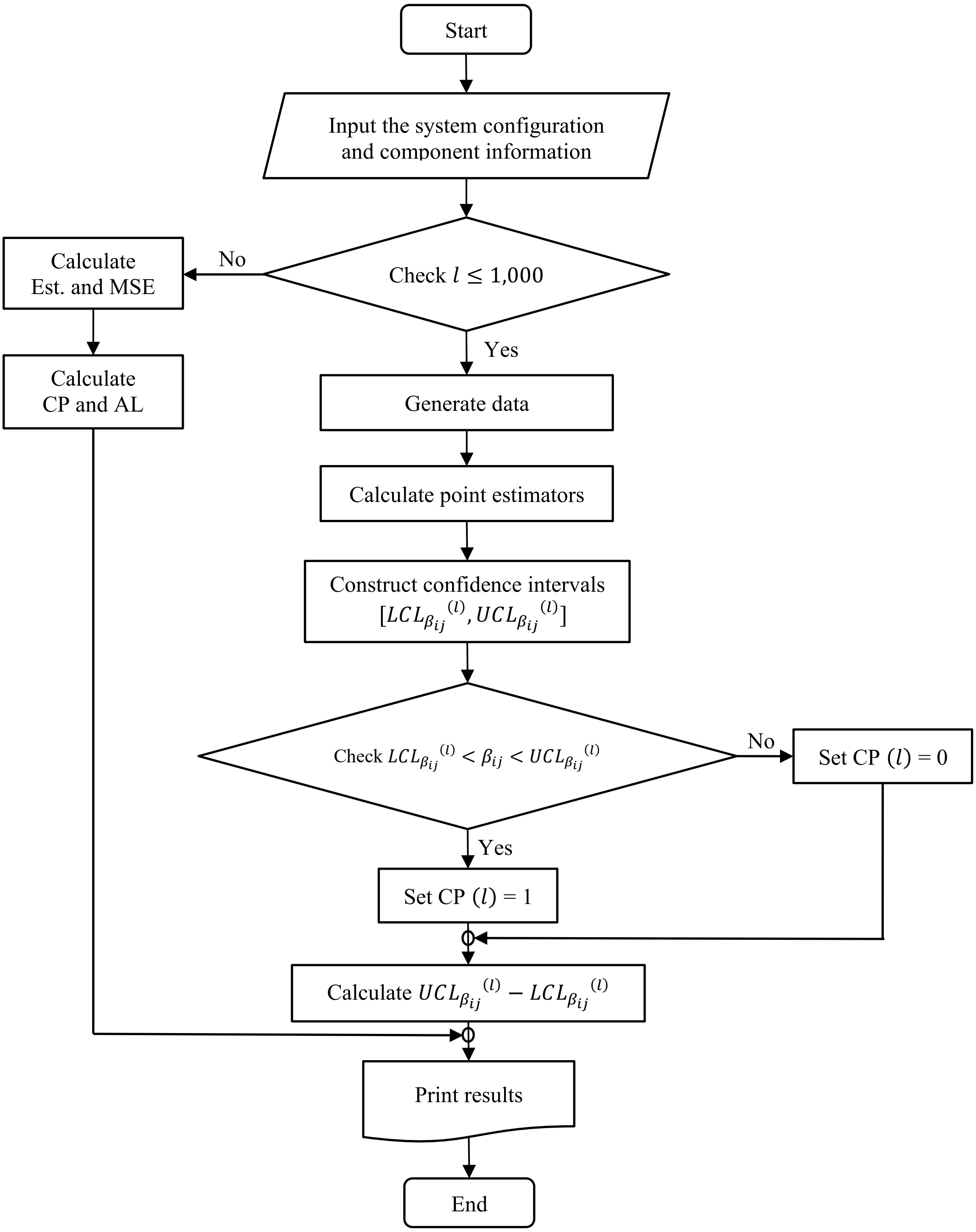

Figure 1. Monte Carlo simulation flowchart

Table 1 reports the parameter estimates (Est.) along with the mean squared error (MSE) for Scenario 1, where both explanatory variables exhibit no multicollinearity. From the results reveal that the parameter estimates (Est.) obtained from all methods closely align with the true parameter values across various sample sizes. The MSE analysis indicates that all estimators exhibit a monotonic trend, where an increase in sample sizes corresponds to a decrease in estimated MSE values. At n = 50, almost all Bayes(N) estimators have the lowest MSE. The majority of Bootstrap and Bayes(N) estimators demonstrate the lowest MSE when n = 100. For n = 200, almost all Bootstrap estimators reveal the lowest MSE. Additionally, it is important to highlight that, regardless of the sample sizes, the MSE of all estimators from all of them exhibits strikingly similar characteristics.

Tables 2 and 3 present the parameter estimates (Est.) and the mean squared error (MSE) results for Scenario 2, in which both explanatory variables demonstrate multicollinearity with correlations of 0.3 and 0.5, respectively. From the results reveal that the parameter estimates (Est.) obtained from all methods closely align with the true parameter values across various sample sizes. The MSE analysis indicates that all estimators exhibit a monotonic trend, where an increase in sample sizes corresponds to a decrease in estimated MSE values. The results indicate that the MSE of the Bayes(N) consistently surpasses that of other methods across nearly all estimators. The Bayes(L) follows closely when n = 50 and 100, whereas at n = 200, the MSE of all methods exhibits similar behavior. Furthermore, the MSE of both the MLE and the Bootstrap techniques exhibit quite similar behavior across all sample sizes.

Tables 4 and 5 present the mean squared error (MSE) results for Scenario 2, where both explanatory variables exhibit multicollinearity with correlations of 0.7 and 0.95, indicating a high level of collinearity. From the results reveal that the parameter estimates (Est.) obtained from all methods closely align with the true parameter values across various sample sizes. The MSE analysis indicates that all estimators exhibit a monotonic trend, where an increase in sample sizes corresponds to a decrease in estimated MSE values. The performance of the Bayes(N) method's MSE clearly surpasses that of alternative approaches in nearly every estimator, while the Bayes(L) method consistently ranks second across all sample sizes. Additionally, the MSE of both the MLE and the Bootstrap techniques exhibit quite comparable behavior across all sample sizes. Furthermore, it is important to observe that almost all estimators obtained using the Bootstrap method exhibit a lower MSE compared to those from the MLE method when the correlation is 0.95 at sample sizes of 100 and 200.

Table 1. Parameter estimates (Est.) and MSE for Scenario 1: $\boldsymbol{\theta} = (0.2, -0.1, 0.1, -0.3, 0.2, 0.1)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
50	0.2	0.1743	**0.0667**	0.1475	0.0684	0.1391	0.0682	0.1356	0.0685
	-0.1	-0.0975	0.0134	-0.0968	0.0137	-0.0948	**0.0130**	-0.0966	0.0133
	0.1	0.1004	**0.0015**	0.1014	**0.0015**	0.1024	**0.0015**	0.1029	**0.0015**
	-0.3	-0.3126	0.0491	-0.3285	0.0494	-0.3088	0.0475	-0.3146	**0.0461**
	0.2	0.1995	**0.0082**	0.1999	0.0083	0.1953	**0.0082**	0.1972	**0.0082**
	0.1	0.0999	0.0010	0.1007	0.0010	0.0980	0.0010	0.0989	**0.0009**
100	0.2	0.1926	0.0330	0.1787	**0.0327**	0.1709	0.0332	0.1713	0.0337
	-0.1	-0.1002	0.0060	-0.1006	0.0060	-0.0991	**0.0059**	-0.1003	**0.0059**
	0.1	0.0999	0.0008	0.1005	**0.0007**	0.1015	**0.0007**	0.1015	**0.0007**
	-0.3	-0.3107	0.0230	-0.3200	**0.0229**	-0.3099	0.0232	-0.3147	**0.0229**
	0.2	0.2005	**0.0039**	0.2007	**0.0039**	0.1983	**0.0039**	0.1995	**0.0039**
	0.1	0.1011	**0.0005**	0.1017	**0.0005**	0.1004	**0.0005**	0.1010	**0.0005**
200	0.2	0.1884	0.0170	0.1810	**0.0169**	0.1749	0.0174	0.1762	0.0175
	-0.1	-0.101	**0.0027**	-0.1011	**0.0027**	-0.1005	0.0028	-0.1014	0.0028
	0.1	0.1004	**0.0004**	0.1007	**0.0004**	0.1015	**0.0004**	0.1014	**0.0004**
	-0.3	-0.3078	0.0113	-0.3124	**0.0111**	-0.3063	0.0116	-0.3095	0.0115
	0.2	0.1989	**0.0016**	0.1989	0.0017	0.1977	0.0017	0.1983	0.0017
	0.1	0.1005	**0.0002**	0.1008	**0.0002**	0.1000	**0.0002**	0.1004	**0.0002**

Note: The boldface identifies the smallest MSE for each case.

Table 2. Parameter estimates (Est.) and MSE for Scenario 2 (correlation level is 0.3): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
50	1.2	1.1877	**0.0432**	1.1750	0.0449	1.1483	0.0453	1.1226	0.0445
	-0.45	-0.4705	0.1134	-0.4765	0.1171	-0.4145	0.1037	-0.3858	**0.0953**
	0.2	0.2070	0.1054	0.2097	0.1084	0.1997	0.0921	0.2170	**0.0859**
	-0.63	-0.662	0.0812	-0.6877	0.0867	-0.6326	0.0662	-0.6135	**0.0535**
	0.21	0.1985	0.1891	0.1948	0.1966	0.1536	0.1487	0.1380	**0.1299**
	0.12	0.1424	0.188	0.1484	0.1939	0.1045	0.1442	0.0920	**0.1229**
100	1.2	1.1856	**0.0208**	1.1787	0.0214	1.1658	0.0216	1.1515	0.0217
	-0.45	-0.4485	0.0542	-0.4502	0.0546	-0.4163	0.0531	-0.4049	**0.0519**
	0.2	0.2055	0.0475	0.2060	0.0482	0.1981	0.0437	0.2113	**0.0430**
	-0.63	-0.6376	0.0405	-0.6504	0.0418	-0.6214	0.0370	-0.6164	**0.0339**
	0.21	0.2045	0.0882	0.2040	0.0899	0.1776	0.0780	0.1742	**0.0745**
	0.12	0.1185	0.0914	0.1206	0.0943	0.0995	0.0787	0.0954	**0.0764**
200	1.2	1.1962	**0.0103**	1.1929	0.0104	1.1878	0.0104	1.1792	0.0105
	-0.45	-0.4537	0.0281	-0.4547	0.0282	-0.4376	0.0281	-0.4331	**0.0269**
	0.2	0.2028	0.0238	0.2032	0.0239	0.1961	**0.0228**	0.2066	0.0229
	-0.63	-0.6344	0.0173	-0.6407	0.0176	-0.6238	0.0164	-0.6231	**0.0159**
	0.21	0.2050	0.0395	0.2047	0.0398	0.1883	**0.0362**	0.1879	0.0369
	0.12	0.1246	0.0411	0.1259	0.0412	0.1131	**0.0372**	0.1128	0.0373

Note: The boldface identifies the smallest MSE for each case.

Table 3. Parameter estimates (Est.) and MSE for Scenario 2 (correlation level is 0.5): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
50	1.2	1.1883	**0.0384**	1.1760	0.0400	1.1534	0.0403	1.1301	0.0401
	-0.45	-0.4711	0.1366	-0.4779	0.1410	-0.4107	0.1184	-0.3781	**0.1072**
	0.2	0.2067	0.1247	0.2090	0.1278	0.1860	0.1033	0.1949	**0.0933**
	-0.63	-0.6602	0.0732	-0.6867	0.0790	-0.6389	0.0629	-0.6186	**0.0508**
	0.21	0.1928	0.2315	0.188	0.2408	0.1535	0.1759	0.1347	**0.1432**
	0.12	0.1445	0.2207	0.1523	0.2300	0.1153	0.1621	0.1058	**0.1320**
100	1.2	1.1863	**0.0184**	1.1798	0.0190	1.1690	0.0193	1.1563	0.0195
	-0.45	-0.4486	0.0636	-0.4505	0.0642	-0.4109	0.0610	-0.4019	**0.0584**
	0.2	0.2043	0.0568	0.2043	0.0574	0.1867	0.0504	0.1985	**0.0498**
	-0.63	-0.6384	0.036	-0.6506	0.0372	-0.6256	0.0334	-0.6189	**0.0305**
	0.21	0.2047	0.1064	0.2033	0.1089	0.1801	0.0894	0.1738	**0.0840**
	0.12	0.1192	0.1095	0.1212	0.1127	0.1047	0.0906	0.1011	**0.0861**
200	1.2	1.1964	**0.0091**	1.1930	0.0093	1.1877	0.0093	1.1807	0.0095
	-0.45	-0.4545	0.0331	-0.4557	0.0333	-0.4338	0.0325	-0.4316	**0.0315**
	0.2	0.2031	0.0283	0.2039	0.0286	0.1923	0.0270	0.2020	**0.0268**
	-0.63	-0.6342	0.0156	-0.6407	0.0159	-0.6263	0.0151	-0.6249	**0.0146**
	0.21	0.2031	0.0473	0.2034	0.0478	0.1879	**0.0422**	0.1879	0.0424
	0.12	0.1261	0.0502	0.1272	0.0504	0.1183	**0.0444**	0.1162	0.0445

Note: The boldface identifies the smallest MSE for each case.

Table 4. Parameter estimates (Est.) and MSE for Scenario 2 (correlation level is 0.7): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
50	1.2	1.1889	**0.0349**	1.1766	0.0366	1.1552	0.0374	1.1342	0.0370
	-0.45	-0.476	0.1945	-0.4828	0.2006	-0.4026	0.1534	-0.3591	**0.1308**
	0.2	0.2101	0.1799	0.2126	0.1844	0.1741	0.1372	0.1684	**0.1143**
	-0.63	-0.6603	0.0677	-0.6858	0.0731	-0.6401	0.0594	-0.6214	**0.0481**
	0.21	0.1863	0.3404	0.1812	0.3482	0.1504	0.2244	0.1342	**0.1626**
	0.12	0.1510	0.3239	0.1578	0.3312	0.1244	0.2081	0.1157	**0.1467**
100	1.2	1.1890	**0.0176**	1.1825	0.0180	1.1723	0.0182	1.1604	0.0188
	-0.45	-0.4564	0.0906	-0.4583	0.0918	-0.4107	0.0809	-0.3955	**0.0750**
	0.2	0.2109	0.0939	0.2110	0.0954	0.1831	0.0802	0.1881	**0.0751**
	-0.63	-0.6559	0.0337	-0.6677	0.0353	-0.6434	0.0316	-0.6371	**0.0283**
	0.21	0.2221	0.1424	0.2208	0.1448	0.1986	0.1091	0.1897	**0.0974**
	0.12	0.1364	0.1412	0.1379	0.1426	0.1230	0.1059	0.1224	**0.0959**
200	1.2	1.1995	**0.0090**	1.1964	0.0091	1.1914	0.0091	1.1857	**0.0090**
	-0.45	-0.4601	0.0459	-0.4615	0.0460	-0.4307	0.0430	-0.4287	**0.0414**
	0.2	0.2089	0.0436	0.2093	0.0439	0.1886	0.0394	0.1958	**0.0389**
	-0.63	-0.6321	0.0147	-0.6381	0.0149	-0.6264	0.0142	-0.6246	**0.0138**
	0.21	0.2116	0.0693	0.2113	0.0696	0.1971	0.0582	0.1962	**0.0579**
	0.12	0.1160	0.0674	0.1167	0.0677	0.1119	0.0564	0.1103	**0.0562**

Note: The boldface identifies the smallest MSE for each case.

Table 5. Parameter estimates (Est.) and MSE for Scenario 2 (correlation level is 0.95): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
50	1.2	1.1889	**0.0315**	1.1768	0.0330	1.1636	0.0341	1.1462	0.0333
	-0.45	-0.4940	0.9213	-0.5067	0.9286	-0.3307	0.3849	-0.2386	**0.1958**
	0.2	0.2277	0.8978	0.2368	0.9036	0.0919	0.3667	0.0328	**0.1706**
	-0.63	-0.6566	0.0609	-0.6818	0.0658	-0.6373	0.0564	-0.6169	**0.0451**
	0.21	0.1424	1.5871	0.1350	1.5180	0.1313	0.4588	0.1258	**0.1556**
	0.12	0.1876	1.5287	0.1975	1.4661	0.1483	0.4328	0.1285	**0.1351**
100	1.2	1.1860	**0.0154**	1.1801	0.0157	1.1737	0.0162	1.1630	0.0166
	-0.45	-0.4552	0.4186	-0.4562	0.4148	-0.3422	0.2485	-0.2814	**0.1679**
	0.2	0.2105	0.4048	0.2093	0.4020	0.1099	0.2336	0.0680	**0.1504**
	-0.63	-0.6395	0.0294	-0.6519	0.0302	-0.6324	0.0288	-0.6211	**0.0258**
	0.21	0.2128	0.7606	0.2107	0.7504	0.1839	0.3315	0.1617	**0.1544**
	0.12	0.1138	0.7565	0.1169	0.7476	0.1198	0.3250	0.1260	**0.1531**
200	1.2	1.1967	**0.0078**	**1.1939**	0.0079	1.1904	0.0081	1.1864	0.0080
	-0.45	-0.4616	0.2133	-0.4675	0.2106	-0.3970	0.1573	-0.3608	**0.1288**
	0.2	0.2087	0.2040	0.2131	0.2014	0.1501	0.1486	0.1210	**0.1212**
	-0.63	-0.6335	0.0129	-0.6399	0.0131	-0.6296	0.0131	-0.6266	**0.0127**
	0.21	0.1903	0.3499	0.1865	0.3470	0.1779	0.2113	0.1654	**0.1433**
	0.12	0.1380	0.3610	0.1429	0.3567	0.1367	0.2165	0.1451	**0.1453**

Note: the boldface identifies the smallest MSE for each case.

Table 6 presents the 95% coverage probability (CP) and the average length (AL) for Scenario 1, in which both explanatory variables show no multicollinearity. The CP for all methods generally approximates the nominal confidence level across different sample sizes. Furthermore, as sample sizes increase, the AL of the 95% confidence intervals decreases for all methods. The AL of the Bayes(N) method was similar to that of the Bayes(L) across all sample sizes, and it was also shorter than the AL of the MLE and Bootstrap methods.

Tables 7 and 9 present the 95% coverage probability (CP) and the average length (AL) for Scenario 2, where explanatory variables exhibit multicollinearity, at correlation values of 0.3, 0.5, and 0.7, respectively. The CP for all methods generally approximates the nominal confidence level across different sample sizes. Furthermore, as sample sizes increase, the AL of the 95% confidence intervals decreases for all methods. When the values are set at n = 50 and 100, the AL of the Bayes(N) method is observed to be the lowest compared to other methods, with the Bayes(L) method following closely behind. In the scenario where n = 200, the AL of the Bayes(N) method was found to be comparable to that of the Bayes(L) across all sample sizes, and it also demonstrated a shorter AL than both the MLE and Bootstrap methods.

Table 10 reports the 95% coverage probability (CP) and the average length (AL) for Scenario 2, which is defined by the presence of multicollinearity among explanatory variables, specifically at correlation values of 0.95. The CP for all methods generally approximates the nominal confidence level across different sample sizes. Furthermore, as sample sizes increase, the AL of the 95% confidence intervals decreases for all methods. The observation indicates that the AL of the Bayes(N) method is the shortest, with the Bayes(L) method following closely behind. Moreover, the AL of the MLE and Bootstrap methods demonstrates remarkably similar behaviors.

Table 6. CP and AL for Scenario 1: $\boldsymbol{\theta}$ = (0.2, -0.1, 0.1, -0.3, 0.2, 0.1).

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		CP	AL	CP	AL	CP	AL	CP	AL
50	0.2	95.4	1.0450	93.7	1.0405	94.9	1.0097	95.3	1.0131
	-0.1	94.0	0.4466	93.6	0.4557	94.2	0.4380	94.2	0.4422
	0.1	94.9	0.1587	94.1	0.1594	93.9	0.1539	94.8	0.1548
	-0.3	96.2	0.8896	94.8	0.8913	95.5	0.8623	95.6	0.8550
	0.2	94.8	0.3450	92.8	0.3569	94.0	0.3410	94.1	0.3403
	0.1	95.7	0.1274	95.0	0.1292	96.0	0.1241	95.2	0.1231
100	0.2	94.7	0.7268	94.8	0.7170	94.6	0.7096	95.2	0.7120
	-0.1	95.9	0.3058	94.6	0.3042	95.1	0.3023	95.3	0.3040
	0.1	95.9	0.1097	93.9	0.1087	94.7	0.1072	95.1	0.1077
	-0.3	95.7	0.6195	95.2	0.6147	95.8	0.6102	95.7	0.6075
	0.2	94.6	0.2336	93.1	0.2362	94.2	0.2326	94.5	0.2318
	0.1	95.5	0.0880	95.5	0.0875	95.4	0.0867	95.2	0.0863
200	0.2	94.5	0.5112	93.6	0.5032	93.8	0.5014	93.7	0.5045
	-0.1	95.9	0.2139	95.1	0.2133	96.2	0.2127	95.5	0.2126
	0.1	94.3	0.0769	93.9	0.0762	93.6	0.0756	95.0	0.0759
	-0.3	95.9	0.4353	95.3	0.4307	95.0	0.4303	95.1	0.4288
	0.2	95.7	0.1625	94.9	0.1620	94.9	0.1615	95.3	0.1615
	0.1	95.0	0.0616	95.6	0.0611	94.2	0.0612	94.8	0.0608

Table 7. CP and AL for Scenario 2 (correlation level is 0.3): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		CP	AL	CP	AL	CP	AL	CP	AL
50	1.2	95.9	1.2994	93.3	1.2998	95.4	1.2507	95.5	1.2127
	-0.45	95.9	1.2936	94.5	1.2951	95.4	1.2346	96.5	1.2112
	0.2	95.2	1.1094	92.6	1.1304	95.8	1.0466	96.6	0.9943
	-0.63	95.6	1.6745	93.0	1.7278	96.2	1.5548	96.0	1.4969
	0.21	94.3	1.6814	93.4	1.7199	95.9	1.5654	96.0	1.5000
	0.12	95.9	1.2994	93.3	1.2998	95.4	1.2507	95.5	1.2127
100	1.2	96.8	0.5750	94.7	0.5707	95.0	0.5669	94.5	0.5603
	-0.45	94.6	0.9015	94.7	0.8990	94.3	0.8813	94.1	0.8706
	0.2	96.5	0.8950	95.1	0.8910	95.5	0.8664	95.9	0.8649
	-0.63	94.6	0.7658	92.8	0.7701	94.7	0.7399	94.1	0.7216
	0.21	94.1	1.1481	93.3	1.1522	94.2	1.0937	94.9	1.0856
	0.12	94.0	1.1490	92.5	1.1530	94.3	1.0938	93.5	1.0768
200	1.2	95.1	0.4019	94.3	0.3999	93.6	0.3970	94.0	0.3956
	-0.45	93.8	0.6297	93.3	0.6235	93.3	0.6222	93.3	0.6163
	0.2	95.2	0.6250	94.5	0.6199	95.1	0.6102	94.8	0.6112
	-0.63	95.4	0.5337	94.5	0.5282	96.0	0.5205	94.9	0.5166
	0.21	95.6	0.7992	95.3	0.7954	95.4	0.7756	95.0	0.7758
	0.12	95.2	0.8005	94.1	0.7979	95.2	0.7713	94.7	0.7759

Table 8. CP and AL for Scenario 2 (correlation level is 0.5): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		CP	AL	CP	AL	CP	AL	CP	AL
50	1.2	94.5	0.7733	93.7	0.7736	93.4	0.7636	94.1	0.7452
	-0.45	96.1	1.4230	93.2	1.4282	95.7	1.3579	95.4	1.3087
	0.2	95.6	1.4159	94.7	1.4212	95.9	1.342	96.2	1.3036
	-0.63	94.7	1.0485	92.6	1.0708	94.5	0.9983	96.1	0.9489
	0.21	95.3	1.8378	91.9	1.8974	95.9	1.6892	96.7	1.5969
	0.12	94.4	1.8401	93.1	1.8827	96.2	1.6841	96.7	1.5977
100	1.2	96.6	0.5411	94.1	0.5376	94.6	0.5347	95.1	0.5299
	-0.45	94.7	0.9853	94.3	0.9814	93.2	0.9565	94.7	0.9410
	0.2	96.0	0.9791	94.9	0.9733	96.6	0.9412	96.1	0.9380
	-0.63	94.7	0.7244	93.2	0.7278	94.6	0.7025	95.0	0.6883
	0.21	94.0	1.2568	93.2	1.2589	95.1	1.1896	94.6	1.1697
	0.12	94.3	1.2572	92.6	1.2595	95.2	1.1917	94.7	1.1698
200	1.2	95.0	0.3783	93.7	0.3762	94.1	0.3750	93.9	0.3728
	-0.45	94.5	0.6878	93.0	0.6808	93.9	0.6752	93.7	0.6696
	0.2	95.4	0.6838	94.9	0.6781	94.9	0.6659	95.2	0.6635
	-0.63	95.6	0.5048	95.0	0.5012	95.3	0.4947	95.8	0.4905
	0.21	95.5	0.8742	94.9	0.8692	95.5	0.8390	95.5	0.8418
	0.12	94.8	0.8756	93.5	0.8729	94.6	0.8372	95.0	0.8414

Table 9. CP and AL for Scenario 2 (correlation level is 0.7): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		CP	AL	CP	AL	CP	AL	CP	AL
50	1.2	94.7	0.7333	93.8	0.7352	94.3	0.7226	94.2	0.7102
	-0.45	95.0	1.7077	93.9	1.7193	96.1	1.5818	96.5	1.4962
	0.2	95.9	1.7011	94.4	1.7091	96.2	1.5574	97.1	1.4929
	-0.63	95.1	0.9992	92.7	1.0223	95.0	0.9604	95.5	0.9105
	0.21	93.9	2.2116	92.1	2.2757	95.4	1.9597	96.7	1.7997
	0.12	94.6	2.2122	92.7	2.2672	96.1	1.9566	96.7	1.7898
100	1.2	95.1	0.5131	93.9	0.5115	93.8	0.5064	93.5	0.5031
	-0.45	95.4	1.1789	93.5	1.1709	95.2	1.1132	95.2	1.0991
	0.2	94.2	1.1765	94.2	1.1658	94.5	1.1089	94.5	1.096
	-0.63	94.3	0.6946	93.1	0.6938	94.2	0.6750	94.6	0.6605
	0.21	95.0	1.5081	93.3	1.4954	96.1	1.3859	96.4	1.3459
	0.12	95.8	1.5133	95.0	1.5133	97.0	1.3892	96.9	1.3531
200	1.2	93.9	0.3589	92.3	0.3559	92.7	0.3537	92.6	0.3548
	-0.45	94.4	0.8238	93.6	0.8113	93.8	0.7970	95.0	0.7905
	0.2	95.4	0.8218	94.4	0.8094	95.2	0.7892	95.1	0.7877
	-0.63	95.6	0.4814	94.5	0.4781	95.6	0.4728	95.2	0.4677
	0.21	95.3	1.0496	94.4	1.0403	95.4	0.9881	96.2	0.9882
	0.12	95.8	1.0508	94.9	1.0373	95.7	0.9898	95.7	0.9854

Table 10. CP and AL for Scenario 2 (correlation level is 0.95): $\boldsymbol{\theta} = (1.2, -0.45, 0.2, -0.63, 0.21, 0.12)$.

n	θ	MLE		Bootstrap		Bayes(L)		Bayes(N)	
		CP	AL	CP	AL	CP	AL	CP	AL
50	1.2	95.4	0.6914	93.5	0.696	91.4	0.6594	90.9	0.6384
	-0.45	96.2	3.8378	93.9	3.8500	97.1	2.8120	97.6	2.2013
	0.2	96.1	3.8367	94.5	3.8429	97.3	2.8087	99.1	2.1904
	-0.63	95.2	0.9472	92.6	0.9745	93.6	0.8834	93.2	0.8278
	0.21	93.4	4.9930	92.9	5.0174	97.4	3.3571	99.0	2.4024
	0.12	94.9	4.9942	94.0	5.0064	97.8	3.346	99.5	2.4189
100	1.2	95.6	0.4837	94.5	0.4844	92.5	0.4627	91.3	0.4575
	-0.45	96.0	2.6453	95.2	2.6095	95.6	2.1221	96.3	1.8667
	0.2	96.3	2.6436	94.5	2.6105	95.7	2.1074	96.9	1.8602
	-0.63	94.9	0.6544	94.2	0.6628	92.8	0.6219	92.1	0.6056
	0.21	95.0	3.3946	92.8	3.3333	96.6	2.5710	98.6	2.1190
	0.12	94.3	3.3934	93.1	3.3255	97.2	2.5678	99.1	2.1244
200	1.2	94.3	0.3376	94.1	0.3374	92.2	0.3243	91.6	0.3235
	-0.45	95.3	1.8477	95.1	1.8169	93.1	1.5645	94.4	1.4741
	0.2	95.3	1.8474	94.7	1.8187	94.5	1.5658	94.9	1.4743
	-0.63	95.2	0.4554	95.3	0.4549	93.8	0.4359	93.2	0.4277
	0.21	94.5	2.3598	93.7	2.3189	95.4	1.9320	96.3	1.7546
	0.12	94.7	2.3611	93.2	2.3188	95.4	1.9240	97.0	1.7482

7- Application

To illustrate the application of the proposed parameter estimations, this study considers two data sets; Allegheny County Crash Data (Data1) [35] and the fatal railway accidents and fatalities in Britain (Data2) [36]. It calculates the parameter estimates and constructs the 95% confidence intervals via the maximum likelihood estimation method, bootstrapping method as well as 95% credible confidence intervals based on Bayesian method under the two prior distributions. Besides, the three information criteria, namely the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and the deviance information criterion (DIC) are applied to compare estimations. All results are reported in Tables 11 and 12.

Data 1: This study considers a dataset of injuries and fatalities from crash reports in Allegheny County, USA, for the years 2019 to 2021, provided by the Pennsylvania Department of Transportation. The dataset included 66 observations. The response variables under consideration were the number of all injuries from the accident (Y_1) and the number of fatalities (Y_2). The explanatory variables included the number of all motor vehicles involved in the accident (X_1) and the posted speed limit in miles per hour (X_2). The variables X_1 and X_2 are not correlated, with a correlation coefficient of -0.0926 (p-value = 0.4592). The chi-square goodness-of-fit test (GOT) for the response variables using the ZTBVP distribution yielded a p-value of 0.4069, suggesting that the data can be modelled by the ZTBVP distribution.

Data 2: It considers a dataset of fatal railway accidents and fatalities in Britain from 1967 to 2003 with 54 observations. The response variables under consideration wrer the number of accidents due to Train/road vehicle collision (Y_1) and the number of fatalities due to Train/road vehicle collision fatalities (Y_2). The explanatory variables considered were the number of movements and non-movement accidents (X_1) and movements and non-movement accidents fatalities (X_2). The correlation coefficient between the explanatory variables X_1 and X_2 stood at 0.9998 (p-value < 0.05). Thus, two variables are correlated. The p-value for the chi-square goodness of fit (GOT) test using the ZTBVP distribution of the response variable was 0.9998. Thus, this data can be modelled by the ZTBVP distribution.

The results from Table 11 suggest that Bayes(N) method provides better fitting than Bayes(L) method in terms of DIC. Regarding the AIC and BIC, all methods exhibit remarkably similar behavior. Moreover, the results of significant explanatory variables that were selected from the two explanatory variables. It was only reported that X_1 show significant for Y_1 in all methods while X_2 is significant for Y_2 in only bootstrap method.

Table 11. Parameter estimates, the 95% confidence intervals (in parentheses), and the three information criteria for Data 1

Parameters	MLE	Bootstrap	Bayes(L)	Bayes(N)
β_{10}	-0.4906 (-1.8579,0.8767)	-0.5511 (-1.9934,0.893)	-0.4859 (-1.7151,0.5689)	-0.4755 (-1.3203,0.6381)
β_{11}	0.4694* (0.1823,0.7564)	0.5225* (0.2467,1.0965)	0.4368* (0.1442,0.6852)	0.4624* (0.2639,0.6568)
β_{12}	-0.0192 (-0.0519,0.0135)	-0.0223 (-0.0699,0.0071)	-0.0190 (-0.0527,0.0152)	-0.0202 (-0.0462,0.0022)
β_{20}	0.0136 (-4.9301,4.9573)	-0.2270 (-2.3306,1.3978)	-0.5566 (-3.8453,2.3075)	-0.2970 (-2.0581,1.4493)
β_{21}	-0.2340 (-1.6878,1.2198)	-0.3375 (-0.8647,0.3182)	-0.5013 (-1.6211,0.4726)	-0.3966 (-1.5967,0.4935)
β_{22}	-0.0694 (-0.2141,0.0754)	-0.0734* (-0.449,-0.0102)	-0.0445 (-0.1542,0.0291)	-0.0558 (-0.1547,0.0213)
AIC	156.2135	157.1081	156.2997	**156.1241**
BIC	169.3514	170.2461	169.4376	**169.2620**
DIC			144.2997	**144.1241**

Note: (*) denotes the 95% confidence interval for the ZTBVP regression model does not contain zero (statistically significant) and the boldface identifies the minimum value of each information criteria.

The results from Table 12 suggest that Bayes(N) method provides better fitting than Bayes(L) method in term of DIC. In terms of AIC and BIC, the Bayes(N) method demonstrates superior fitting compared to all other methods, with Bayes(L) closely trailing. Meanwhile, the Bootstrap method outperforms the MLE methods in fitting quality. Moreover, the results of significant explanatory variables that were selected from the three explanatory variables. It was only reports that X_2 show significant for Y_2 in Bayes(L) and Bayes(N) methods. Moreover, it was found that the MLE method was unable to estimate confidence intervals at the 95% confidence level. This may be due to the extremely high collinearity among the explanatory variables, thus results in the inability to compute the inverse of the observed Fisher information.

Table 12. Parameter estimates, the 95% confidence intervals (in parentheses), and the three information criteria for Data 2

Parameters	MLE	Bootstrap	Bayes(L)	Bayes(N)
β_{10}	0.7927	0.7276* (0.1216,1.3756)	0.7507* (0.7136,0.8058)	0.7914* (0.7710,0.8143)
β_{11}	0.0263	-0.0028 (-0.1770,0.1535)	0.0148 (-0.0265,0.0389)	-0.0088 (-0.0328,0.0092)
β_{12}	-0.0200	0.0091 (-0.1453,0.1829)	-0.0085 (-0.032,0.0321)	0.0146 (-0.0030,0.0383)
β_{20}	0.0215	-0.0714 (-0.4789,0.3752)	0.0223 (-0.007,0.0512)	-0.0120 (-0.0494,0.0367)
β_{21}	-0.1240	-0.0260 (-0.1360,0.1085)	-0.0530 (-0.0917,-0.0135)	-0.0474 (-0.0905,-0.0277)
β_{22}	0.1234	0.0274 (-0.1059,0.1360)	0.0537* (0.0148,0.0921)	0.0483* (0.0290,0.0909)
AIC	545.9418	530.4944	528.5201	**524.4709**
BIC	557.8757	542.4283	540.4540	**536.4048**
DIC			516.5201	**512.4709**

Note: (*) denotes the 95% confidence interval for the ZTBVP regression model with multicollinearity does not contain zero (statistically significant) and the boldface identifies the minimum value of each information criteria.

8- Discussion

In statistics, classical and Bayesian inference differ significantly. One classical approach that is commonly employed in count regression models is the maximum likelihood method. For an asymptotically large sample size of data, the maximum likelihood estimate of parameters is valid, according to Wang et al. (2023) [12] and Kummaraka & Srisuradetchai (2023) [13]. Insufficient sample sizes can cause the maximum likelihood estimates to become unstable and show significant standard errors of the estimates, which has a detrimental effect on statistical inference. Chowdhury & Islam (2016) [11] introduced the maximum likelihood method for parameter estimation of the zero-truncated bivariate Poisson regression model. The other estimates in this model have yet to be discovered. The multicollinearity issue in regression models is addressed by the current bootstrapping techniques [19, 20].

As an alternative, the Bayesian technique incorporates empirical knowledge from the likelihood function as well as prior knowledge about the parameters from the prior probability distribution. Therefore, the defined prior distribution determines how well the Bayesian estimation performs. By using a Monte Carlo simulation, performance and sensitivity assessments are carried out to evaluate the Bayesian estimators' resilience to various prior distributions for every model, including estimations of the parameters, bias, mean square error, coverage probability, and average length. Along with the traceplot, autocorrelation for sampled values and posterior densities, as well as AIC, BIC, and DIC. In order to undertake performance assessments, this paper simply estimated the estimates of the parameters, mean square error, coverage probability, and average length for simulation studies, while AIC, BIC, and DIC were used for data applications.

This study presents the Bayesian estimation for the zero-truncated bivariate Poisson regression model, employing the Metropolis-Hastings algorithm and utilizing two prior distributions for the regression coefficients: Laplace and normal distributions. Furthermore, the bootstrap method was proposed. The performance of the Bayes estimators was evaluated alongside the bootstrap and the maximum likelihood estimators. The findings in this study align with those of Choe et al. (2012) [22] and Chaiprasithikul & Duangsaphon [26], indicating that the normal prior distribution consistently outperforms the other methods, with the Laplace prior distribution method closely trailing behind. Nonetheless, certain computational challenges arise when applying the Bayesian approach, particularly regarding the selection of hyperparameter values for prior distributions, which can influence the parameter estimates. Additionally, the bootstrap estimators demonstrated strong performance when the explanatory variables were generated under conditions of multicollinearity, similar to the findings of Sudjai & Duangsaphon (2020) [19] and Perveen & Suhail (2021) [20].

9- Conclusion

This paper provides the Bayesian estimation for the zero-truncated bivariate Poisson model, utilizing two distinct prior distributions: the normal prior and the Laplace prior. Under the squared error loss function, it is apparent that there is no explicit form of the posterior distribution. Consequently, the random walk Metropolis-Hastings algorithm is employed to estimate Bayes estimators. Additionally, a comparison is made between the maximum likelihood estimation and the bootstrap methods. The performance of all methods is evaluated through Monte Carlo simulation, focusing on the estimates of the parameters, the mean square error, the coverage probability, and average length criteria. The criteria are computed for various sample sizes, considering both scenarios of no multicollinearity and the presence of multicollinearity. According to the measures of accuracy for the estimators, the Bayesian method utilizing a normal prior distribution demonstrates superior performance compared to alternative approaches, with the Laplace prior distribution following closely behind. Furthermore, the Bayesian approaches derived from both prior distributions exhibit comparable behavior when the sample size is substantial. Additionally, the bootstrap estimators demonstrate effective performance for high levels of collinearity, particularly when the sample size is substantial.

Two real data sets from the literature are used to illustrate how this model is implemented and to compare the proposed methods using the Akaike information criterion, the Bayesian information criterion, and the deviance information criterion. As a result, Bayesian estimation using a normal prior distribution is superior to other approaches. In addition, the bootstrap method performs better than maximum likelihood estimation under a high level of collinearity of explanatory variables. These results show that the simulated study and the real data analysis are consistent. Therefore, the Bayesian method utilizing a normal prior distribution is recommended for the zero-truncated bivariate Poisson regression model.

10- Declarations

10-1-Author Contributions

Conceptualization, P.R. and M.D.; methodology, P.R. and M.D.; formal analysis, P.R. and M.D.; writing—original draft preparation, P.R. and M.D.; writing—review and editing, P.R. and M.D.; funding acquisition, M.D. All authors have read and agreed to the published version of the manuscript.

10-2-Data Availability Statement

The data presented in this study are available on request from the corresponding author.

10-3-Funding

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10-4-Institutional Review Board Statement

Not applicable.

10-5-Informed Consent Statement

Not applicable.

10-6-Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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