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Innovative Approach to the Optimal Distribution of Citizens' Pension Savings to Non-State Pension Funds

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Abstract

In the Russian Federation, persistent economic and legal tensions surround the allocation of citizens' pension savings; however, individuals retain the option to select the organization that manages the funded portion of their pension. This study aims to address the challenges posed by dynamic programming regarding the optimal distribution of Russian citizens' pension savings to non-state pension funds (NPFs), using predictive analyses of expected returns generated by the Verhulst forecasting equation. The research methodology encompassed system analysis, the Verhulst prognostic equation, dynamic programming models, and conditional optimization based on R. Bellman's equations. The study's information and empirical foundation comprised current regulatory legal acts, data from the Federal State Statistics Service (Rosstat), open data from the Central Bank of the Russian Federation (Bank of Russia), analysis of information sources on the activities of domestic NPFs, results of empirical studies by domestic and foreign authors, and information obtained from open sources on the profitability of 22 NPFs of the Russian Federation. The forecast for the period from 2024 to 2063, using the Verhulst forecasting model developed in this study, indicates that the highest value of expected profitability in 2063, specifically 11.66% in annual terms, should be anticipated from the JSC NPF Alliance, while the minimum (3.54% per annum) is expected from the JSC MNPF BOLSHOY. The solution to the dynamic programming problem concerning the optimal distribution of citizens' pension savings in NPFs demonstrated that the maximum return on investment of pension funds would be achieved under the condition that from 2024 to 2043, funds are invested in JSC NPF FUTURE, and from 2044 to 2063, the funds are invested in the JSC NPF Alliance. The total return on pension savings for the entire investment period (40 years) amounts to 5202%, or more than 52 times the initial investment.

Keywords:

Forecasting;
Verhulst Equation;
Logistic Equation;
Profitability;
R. Bellman Equation;
Conditional Optimization;
Funded Pension.

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1- Introduction

As a mechanism, the state cannot function effectively without crucial organizations such as pension funds. Pension funds are among the most significant institutions in all countries. Both state pension funds and non-state pension funds (NPF) exist [1]. Citizens have a vested interest in selecting non-state organizations for their pension contributions [2]. Consequently, to make an informed choice regarding NPF, individuals require knowledge of each NPF's annual profitability indicator, its fluctuations over the period under consideration, and the factors contributing to these changes. The variation in the annual rate of return of any pension fund serves as a critical criterion for assessing the population's welfare, and by extension, the overall economic condition of the state. Therefore, changes in pension funds' annual returns indicate whether the state can anticipate a positive or negative trend in further economic development [3, 4]. On the topic of this study, 80 scientific articles by domestic and foreign researchers and specialists were analyzed, which can be divided into the following 7 groups:

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- Autoregressive moving-average model, ARIMA (20 articles);
- Autoregressive Conditional Heteroscedasticity, ARCH (18 papers);
- Vector Auto Regression, VAR (7 articles);
- Machine learning, ML (14 papers);
- Neural network, NN (12 research articles);
- Deep learning, DL (9 research articles);
- Logistic equation, LE (8 research articles).

The relative weight of each group in the literature review is shown in Figure 1.

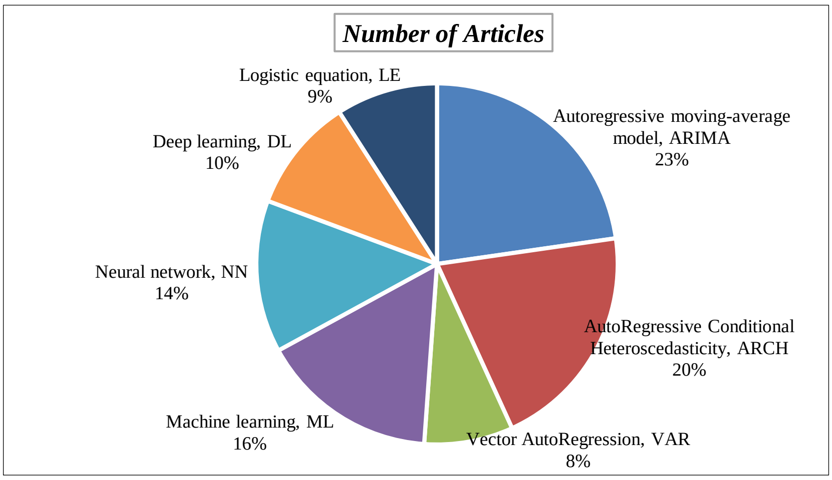

Figure 1. Distribution of articles by research topics

A comparative analysis of the models and methods used for forecasting, their detailed descriptions, and the degree of development of the research problem and its analysis are presented in Table 1.

Table 1. Degree of the problem development of the present study

No.	Method Name	Method description
1	Autoregressive Moving-Average Model, ARIMA	ARIMA – The integrated autoregressive moving average model is a model and methodology for analyzing time series [5-7]. It is an extension of ARMA models for non-stationary time series, which can be made stationary by taking differences of some order from the original time series (so-called integrated or difference-stationary time series) [8-10]. The ARIMA approach to time series is to first assess the stationarity of the series [11-13]. Various tests reveal the presence of unit roots and the order of integration of the time series (usually limited to the first or second order) [14-16]. Then, if necessary (if the order of integration is greater than zero), the series is transformed by taking the difference of the corresponding order and some ARMA model is built for the transformed model [17-20], since it is assumed that the obtained process is stationary, in contrast to the initial nonstationary process (difference-stationary or integrated process of order *d*) [21-24].
2	Autoregressive Conditional Heteroscedasticity, ARCH	Autoregressive conditional heteroskedasticity is a model used in econometrics to analyze time series (primarily financial) [25-27], in which the conditional (on past values of the series) variance of the series depends on past values of the series, past values of these variances and other factors [28]. These models are designed to “explain” the clustering of volatility in financial markets, when periods of high volatility last for some time, followed by periods of low volatility [29, 30]. Moreover, the average (long-term, unconditional) volatility can be considered relatively stable [31-33]. ARCH models were first proposed by Robert Engle in 1982 [34-36]. Already in 1986, Bollerslev proposed a generalization of these models (GARCH) [37-39]. Later, various authors proposed other variants of this type of models that consider certain features [40, 41].
3	Vector Autoregression, VAR	Vector autoregression is a model of the dynamics of several time series in which the current values of these series depend on the past values of the same time series [42-44]. The model was proposed by Christopher Sims as an alternative to simultaneous equation systems, which involve significant theoretical limitations [45, 46]. VAR models are free from the limitations of structural models [47]. Nevertheless, the problem with VAR models is the sharp increase in the number of parameters as the number of time series analyzed and the number of lags increases [48].
4	Machine Learning, ML	Machine learning is a subset of artificial intelligence that utilizes techniques (e.g., deep learning) that allow computers to use experience to improve at problem solving [49-51]. The learning process is based on the following steps [52-54]. 1) Transferring data to the algorithm (In this step, additional information can be transferred to the model, e.g., by obtaining additional data) [55-57]. 2) This data is used to train the model [58]. 3) Testing and deploying the model [59]. 4) Using the deployed model for automated prediction-based problem solving [60-62].
5	Neural Network, NN	Neural network is a type of machine learning in which a computer program mimics the operation of the human brain [63-65]. Just as neurons in the brain transmit signals to each other, in a neural network, computational elements exchange information [66-68]. Each neural network consists of artificial neurons that imitate the work of human neurons [69]. These are software modules or nodes that interact and exchange information to solve a problem [70]. A basic neural network contains three layers of artificial neurons: 1) input — processes external information, analyzes or classifies it and passes it to the next layer; 2) hidden (there may be several) - analyzes the output data of the previous layer, processes it and passes it to the next layer; 3) output — produces the final result after processing all the data [71].
6	Deep Learning, DL	Deep or deep learning is a kind of machine learning based on artificial neural networks [72, 73]. The learning process is called deep learning because the structure of artificial neural networks consists of several input, output and hidden layers [74]. Each layer contains units that transform input data into information that the next layer can use for a particular prediction task [75]. Due to this structure, the computer can learn through its own data processing [76].
7	Logistic Equation, LE	The logistic equation, also known as the Verhulst equation (after the Belgian mathematician who first formulated it), originally appeared in the study of population changes [77-79]. A prognostic model, from the point of view of mathematical economics, is a well-founded assumption about what the state of the object under study will be in the future period [2, 4]. Such a model is built on the basis of probabilistic forecasts [80-82].

Forecasting models from the point of view of mathematical economics represent a reasonable assumption regarding the state of the object under study in the future [6, 24, 44]. The analysis shows that the first six methods, autoregressive integrated moving average (ARIMA) model, vector autoregression (VAR), autoregressive conditional heteroskedasticity (GARCH model), machine learning, neural network, and Deep Learning, are often used to forecast the prices of various financial indicators [7, 23, 52]. These models were built based on probabilistic forecasts. The first three models, ARIMA, VAR, and GARCH, are often used to forecast interest rates, but they have a number of significant drawbacks, including complexity of use and high forecast error [7, 25, 49].

The three methods of forecasting, ML, NN, and DL, work owing to artificial intelligence, the principle of which is complex; therefore, to develop a system that has the ability to make forecasts using any of them, it is necessary to use a special system to which not everyone has access [48, 62, 72]. In addition, the disadvantage of forecasting using artificial intelligence is the inability to verify the calculations with the help of which forecasting results are obtained [50, 63, 74].

Thus, the analysis of the degree of problem development in the present study, shown in Table 1, demonstrated that, at present, information on the expected average annual rate of return of pension funds in the Russian Federation is insufficient because of the significant number and variety of factors affecting it. In this regard, there are significant difficulties in constructing forecasts and their practical applications [83].

There is a great variety of methods for forecasting economic indicators. Most of these methods are characterized by high labor intensity and complex calculations. All of the above indicates that in the modern world, there is no easy, fast, and high-quality method for forecasting the interest rate of return. Simultaneously, the logistic equation originally created for population forecasting can also be used to forecast the NPF pension savings yield. The purpose of this study is to develop a predictive model for estimating the expected annual profitability of NPFs in the Russian Federation based on the Verhulst equation and, considering the obtained results, solve the dynamic programming problem of optimal distribution of pension savings in NPFs.

2- Research Methodology

The Verhulst equation, which has been applied in demographic forecasts, is the optimal solution for forecasting the expected annual return of the Russian Federation NPF [2, 4]. The Verhulst equation is characterized by its simplicity of use, and since this model has proven itself very well in demographic forecasts, there is reason to believe that its use for forecasting the expected annual return of the Russian Federation NPF will also be very effective.

In this study, the Verhulst equation (logistic growth equation) had the following form [4]:

$$\frac{dx(t)}{dt} \cdot \frac{1}{x} = r - \frac{r}{K} \cdot x(t) \tag{1}$$

where x is the value of the annual percentage rate of return of the Russian Federation NPF (%); r is the intrinsic growth rate of the annual interest rate of return, fractions of units; K is the supporting capacity, the maximum possible annual return, % [2, 4].

The exact solution of differential Equation 1 is a logistic function, with an s-shaped curve (logistic curve) of the following form:

$$N(t) = \frac{N_p \cdot N_0 \cdot e^{r \cdot t}}{N_p - N_0 + N_0 \cdot e^{r \cdot t}} \tag{2}$$

where N_p is an indicator, which when using this formula to calculate the population forecast is called N_{max} and denotes the maximum possible number of populations and persons. In this study, this indicator cannot denote the maximum possible annual return because of the high volatility of the annual returns of the NPFs considered. In connection with the above, N_p is an indicator that denotes the auxiliary value of the yield, which is necessary to build a forecast of the annual rate of return of the considered NPFs, %; N_0 is the annual actual yield at the initial point in time (%); t is the number of modeling periods and the number of years of forecasting [2-4].

The auxiliary value of the annual rate of return N_p was calculated using the following formula:

$$N_p = N_2 \frac{N_1 \cdot N_2 + N_2 \cdot N_3 - 2 \cdot N_1 \cdot N_3}{N_2^2 - N_1 \cdot N_3} \tag{3}$$

where N_1, N_2 and N_3 are the values of the annual yield at times t_1, t_2 and t_3, respectively (%).

It is recommended that the intrinsic growth rate of the annualized return r be determined based on the following relationship:

$$r = \frac{\ln N_1 - \ln N_0}{t_1 - t_0} \tag{4}$$

Problem statement: The optimal distribution of pension savings of citizens to the following NPFs of the Russian Federation is planned: JSC MNPF AQUILON, JSC NPF Almaznaya Osen, JSC NPF Alliance, JSC NPF BOLSHOY, JSC NPF Volga-Capital, JSC NPF VTB Pension Fund, JSC NPF GAZFOND Pension Savings, JSC NPF Gefest, JSC National NPF, JSC NPF Doverie, JSC NPF OPF named after V.V. Livanov, JSC NPF First Industrial Alliance, JSC NPF PERSPEKTIVA, JSC NPF Professional, JSC NPF Sberbank, JSC NPF Sotsium, JSC NPF Transneft, JSC Khanty-Mansiysk NPF, JSC NPF Future, JSC NPF Otkritie, JSC NPF Surgutneftegaz, JSC NPF Evolyuciya (a total of 22 Russian NPFs) for the period of labor activity of 40 years, which is determined by the period of labor activity, accepted in a number of scientific studies devoted to progressive technologies of financing the health care of the Russian Federation on the basis of medical savings accounts and pension provision of Russian citizens with the use of PPP [1, 3], i.e. for the period from 2024 to 2063 inclusive. The initial funds S0 are determined by the accumulated amount in the personalized pension account for the period of labor activity. Investments can be made no more than once every five years because if a citizen intends to change the insurer (to move from the Pension Fund of Russia (PFR) to NPF, from NPF to PFR, or from NPF to NPF) more often than once in five years, then, as shown in [1], he loses the investment yield received by the previous insurer. The amount of investment in the selected NPF is a multiple of Δx rubles and depends on the expected annual return. Funds Δx directed to each NPF are returned with the corresponding return $FR_k(x)$, which is then directed again to the selected NPF for the accumulation of pension funds.

The task is to determine the amount of funds allocated for investment in each NPF to maximize the total economic effect. Construction of R. Bellman equations (conditional optimization) [83]. Let x_k denote the amount of funds allocated to the k-th; then, the total economic effect from investments in NPFs of the Russian Federation (objective function) is equal to:

$$Z = \sum_{k=1}^{n} \mathrm{FR}_k(x_k) \tag{5}$$

where n is the total number of NPFs (we consider 22 NPFs, so $n = 22$ for our problem), and k is the number of NPFs ($k = n - 1, n - 2, ..., 2, 1$). Variable x satisfies the following constraints:

$$\sum_{k=1}^{n} x_k = s_0 \tag{6}$$

$$x_k \geq 0, k = 1, 2, \dots, n. \tag{7}$$

It is necessary to find variables $x_1, x_2, \dots, x_n$ satisfying constraint systems 6–7 and maximizing function 5. The equations of states in the considered dynamic programming problem have the following form:

$$s_k = s_{k-1} - x_k, k = 1, 2, \dots, n. \tag{8}$$

where s_k is the state parameter that determines the amount of funds remaining after the k-th step, that is, the amount of funds to be distributed among the remaining n-k NPFs.

Let us introduce the function $Z_k^*(s_{k-1})$ as the conditional optimal economic effect obtained from $k, k+1, \dots, n$-th NPF if funds are distributed among them in an optimal way s_{k-1} ($0 \leq s_{k-1} \leq s_0$). The admissible controls at the k-th step satisfy the condition $0 \leq x_k \leq s_{k-1}$ (either we allocate nothing to the k-th NPF, $x_k = 0$, or no more than what we have by the k-th step, $x_k \leq s_{k-1}$). In view of the above, the R. Bellman equations for this problem have the following form:

$$Z_n^*(s_{n-1}) = \max_{\{x_n\}} FR_n\,(s_{n-1}, x_n) \tag{9}$$

$$Z_k^*(s_{k-1}) = \max_{\{x_k\}} \{FR(s_{k-1}, x_k) + Z_{k+1}^*(s_k)\} \tag{10}$$

These are recurrence relations that allow one to determine the previous value of a function by knowing the subsequent values.

Thus, as follows from the above and the analysis of Equations 1 to 4 and 5 to 10, the theoretical and methodological basis of the study was formed by the publications of domestic and foreign specialists in the field of methodology of economic and mathematical modeling and decision-making, namely dynamic programming (solving the problem of distributing investments between NPFs over several years) and linear programming as one of the possible methods for the optimal distribution of investments between responsibility centers (solving the problem of dynamic programming in the MS Excel software environment as a transport problem), systems analysis, financial econometrics and building forecasting models, mathematical statistics and analysis of results, and an information approach to the analysis of systems. The information and empirical basis of the study were formed by the statutes in place, official information resources such as the official websites of the Central Bank of the Russian Federation and the Social Fund of Russia, and information obtained from open sources.

The analysis in Table 1 shows the main advantages and disadvantages of various forecasting methods and models and the substantiation for selecting a logistic curve (Verhulst equation) to forecast the expected annual return on investment of citizens' pension savings in NPFs. In addition, the applied equations of Bellman (9) and (10), known in the literature as conditional optimization [84], also increase the validity and quality of managerial decisions and provide instrumental support for such decisions.

Figure 2 shows an operational flowchart of a comprehensive system for the optimal distribution of citizens' pension savings in NPFs, which reflects in detail the main aspects and criteria for making key managerial decisions on investing assets.

The algorithm for a comprehensive system for the optimal distribution of citizens' pension assets in NPFs:

Step 1. Enter the initial data required to build a forecasting model using the Verhulst equation: In this step, the initial forecasting data obtained from open sources, for example, [85, 86], are entered into the comprehensive system, which is a Verhulst equation-based mathematical model, algorithm, and software based on the MS Excel software product. The initial forecasting data include the average annual profitability of the NPF over the past several years; the frequency of contracts with the NPF to invest pension savings without loss of profitability that is acceptable for citizens; the acceptable number of such contracts concluded by a citizen with several NPFs at once; the investment period; the investment amount; the share of citizens' pension savings that are subject to insurance when investing in the NPF; and other legal, social, economic, financial, and moral issues that significantly impact the use of the Verhulst equation to forecast the expected annual profitability of the NPF and the possibility of investing in the pension savings of these NPFs.

Step 2. Analyze initial data obtained from open sources: In this step, the quality, completeness, reliability, and verifiability of the available information on the average annual return on investment in citizens' accumulated pension assets in NPFs were assessed. If the obtained data met the specified criteria, we adjusted the initial data to the conditions of the pre-forecast period and proceeded to the next step of the algorithm. Otherwise, we adjust the data in accordance with the identified inaccuracies and repeat Step 1 until the criteria are satisfied.

Step 3. Determine the forecast period duration based on the quality assessment and size of the retrospective data sample: In this step, the forecast period duration is determined based on the following factors: the duration of the investment period determined by the average period of employment during which citizens can afford to invest their pension savings in an NPF to increase their pension income and the size of their pension after the end of their employment, and to ensure more comfortable living conditions for their post-retirement survival period (the period between a citizen's retirement and death); acceptable accuracy of the forecasting model; and volume and quality of the initial forecasting data using the Verhulst equation.

Step 4. Determine the acceptable accuracy of the forecasting model: As a rule, the accuracy of the forecasting model is determined by the quality of the initial data and size and duration of the forecast period. In expert-level models, the error determined on the test data should not exceed 5%, whereas in higher-level models, the error should not exceed 1%. In this step, the forecast period duration is determined based on the following factors: the investment period duration, determined by the average period of employment, during which citizens can afford to invest their pension savings in an NPF to increase their pension income and the size of their pensions after their employment is completed, and to provide themselves with more comfortable living conditions during their post-retirement survival period (the period between a citizen's retirement and death); acceptable accuracy of the forecasting model; and the size and quality of the initial forecasting data using the Verhulst equation.

***Step 5. Build a forecasting model using the Verhulst equation*:** Using Equations 1 to 4, a forecasting model is built, for which the auxiliary value of the annual rate of return N_p is determined using Equation 3 and the intrinsic growth rate of the annualized return r is calculated using Equation 4. Subsequently, the differential equation of logistic growth (1) is solved, the exact solution of which is the logistic function or *s*-shaped curve (logistic curve), represented by Equation 2.

Step 6. Check the quality and accuracy of the constructed forecasting model on the test sample data using the following formula to estimate the average approximation error;

$$\bar{E} = \frac{1}{n} \cdot \sum_{t=1}^{n} \left| \frac{\widehat{y_t} - y_t}{y_t} \right| \cdot 100\% \tag{11}$$

is the value of the annualized return on citizens' pension investments in NPFs, calculated using the Verhulst equation in the *t*-th investment period (in the *t*-th year), %; y_t is the actual value of the annualized return on citizens' pension investments in NPFs, based on the test sample data in the *t*-th investment period (in the *t*-th year), %; n is the total number of investment periods.

If the average approximation error is obtained using Equation 11 corresponds to the specified parameters, we proceed to the next step in the algorithm.

Otherwise, it is necessary to change the forecasting model parameters, such as the intrinsic growth rate of the annualized interest rate of return (r, see Equations 1 and 4), supporting capacity (K, Equation 1), initial time t_0 and the number of forecasting periods. If necessary, it is also possible to increase the acceptable accuracy of modeling $\bar{E}$. Next, we return to Step 5 of this algorithm for a comprehensive system for the optimal distribution of citizens' pension assets in NPFs.

Step 7. Apply the Verhulst equation to forecast the expected annual return on citizens' pension investments in NPFs for the forecast period. Based on the initial data (see Tables 2 and 3), forecasting was conducted using the Verhulst equation (Equations 1 to 4). The forecasting results are presented in Table 4.

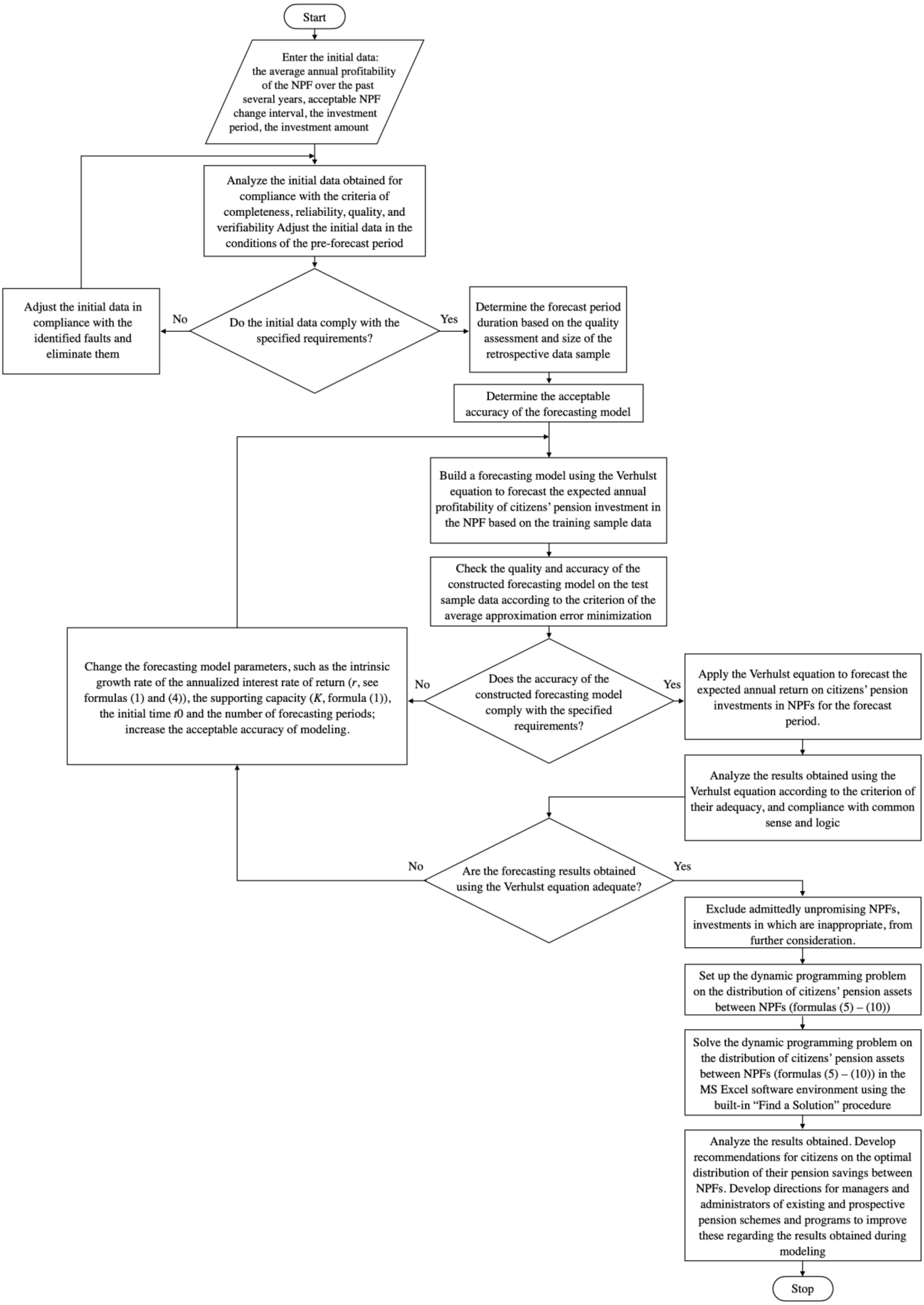

Figure 2. An operations flowchart of a comprehensive system for the optimal distribution of citizens' pension assets in NPFs

Table 2. Annualized return on citizens' pension savings in Russian NPFs according to the Bank of Russia's data (2008-2015)

No.	NPF	Year							
		2008	2009	2010	2011	2012	2013	2014	2015
1	JSC MNPF AQUILON	-35.99%	55.72%	11.40%	-4.20%	3.98%	5.27%	6.69%	8.40%
2	JSC NPF Almaznaya Osen	-25.20%	24.17%	10.99%	1.74%	7.05%	5.03%	2.02%	13.59%
3	JSC NPF Alliance	n/a	n/a	n/a	-4.28%	6.37%	9.84%	-0.13%	12.15%
4	JSC MNPF BOLSHOI	-35.23%	22.80%	10.40%	1.53%	7.13%	7.25%	6.15%	11.33%
5	JSC NPF Volga-Capital	-44.00%	19.00%	13.92%	0.13%	7.35%	8.96%	7.59%	13.69%
6	JSC NPF VTB Pension Fund	-11.15%	17.42%	11.54%	0.73%	10.99%	6.59%	4.72%	10.76%
7	JSC NPF GAZFOND Pension Accumulation JSC	-26.86%	29.35%	11.84%	1.23%	9.17%	7.23%	2.85%	13.92%
8	JSC NPF Gefest	-71.26%	24.59%	16.43%	-1.99%	6.62%	7.37%	0.62%	16.65%
9	JSC National NPF	-0.03%	30.38%	12.29%	1.65%	5.21%	4.94%	1.65%	11.90%
10	JSC NPF Doverie	1.20%	35.70%	11.60%	-3.29%	6.07%	8.35%	5.43%	9.60%
11	JSC NPF OPF named after V.V. Livanov	50.29%	33.37%	13.72%	-5.56%	6.13%	7.51%	6.48%	13.06%
12	JSC NPF First Industrial Alliance	-24.25%	24.25%	14.75%	2.77%	8.35%	8.68%	2.73%	13.34%
13	JSC NPF PERSPECTIVA	-36.46%	46.17%	11.27%	0.34%	6.80%	6.07%	2.19%	12.75%
14	JSC NPF Professionalniy	0.13%	13.46%	6.68%	3.90%	7.07%	7.40%	3.27%	8.11%
15	JSC NPF Sberbank	-27.34%	32.22%	11.31%	-0.09%	7.84%	6.95%	2.67%	10.70%
16	JSC NPF Sotsium	-15.47%	14.41%	11.88%	1.87%	10.24%	8.53%	7.10%	12.43%
17	JSC NPF Transneft	-19.47%	20.33%	13.42%	1.74%	7.16%	6.65%	2.38%	12.97%
18	JSC Khanty-Mansiysk NPF	-15.72%	23.29%	14.85%	2.39%	7.91%	6.61%	0.44%	15.84%
19	JSC NPF Future	n/a	n/a	n/a	n/a	n/a	n/a	1.47%	5.58%
20	JSC NPF Otkritie	n/a	n/a	n/a	n/a	n/a	n/a	8.95%	8.96%
21	JSC NPF Surgutneftegaz	n/a	n/a	n/a	n/a	n/a	n/a	0.25%	12.94%
22	JSC NPF Evolyuciya	n/a	n/a	n/a	n/a	n/a	n/a	7.26%	10.57%

Note: n/a - no data available.

Table 2 (Continued). Annualized return on citizens' pension savings in Russian NPFs according to the Bank of Russia's data (2016-2023)

No.	NPF	Year							
		2016	2017	2018	2019	2020	2021	2022	2023
1	JSC MNPF AQUILON	11.51%	9.71%	6.83%	10.75%	5.17%	2.41%	5.21%	6.39%
2	JSC NPF Almaznaya Osen	12.62%	11.16%	6.78%	8.78%	5.84%	1.31%	7.89%	6.93%
3	JSC NPF Alliance	9.76%	8.83%	4.20%	9.68%	6.42%	1.05%	6.06%	1.13%
4	JSC MNPF BOLSHOI	9.60%	7.16%	2.21%	9.62%	6.33%	2.73%	7.01%	9.54%
5	JSC NPF Volga-Capital	11.31%	9.66%	4.10%	10.81%	5.88%	2.18%	6.79%	4.19%
6	JSC NPF VTB Pension Fund	10.30%	9.02%	5.53%	8.58%	5.96%	3.73%	3.33%	7.20%
7	JSC NPF GAZFOND Pension Accumulation JSC	13.16%	9.53%	6.37%	6.21%	5.46%	5.39%	2.22%	12.90%
8	JSC NPF Gefest	11.95%	9.97%	3.94%	12.16%	5.14%	3.43%	1.20%	3.91%
9	JSC National NPF	11.48%	9.81%	4.08%	9.49%	5.61%	1.61%	5.66%	5.23%
10	JSC NPF Doverie	10.10%	8.49%	4.30%	7.28%	5.41%	3.84%	6.53%	8.00%
11	JSC NPF OPF named after V.V. Livanov	12.31%	8.62%	6.31%	9.33%	6.78%	3.30%	6.24%	9.36%
12	JSC NPF First Industrial Alliance	12.22%	8.14%	5.04%	10.72%	5.96%	3.24%	4.97%	5.36%
13	JSC NPF PERSPECTIVA	10.36%	9.19%	4.78%	9.88%	5.95%	2.33%	6.84%	8.70%
14	JSC NPF Professionalniy	8.25%	8.59%	3.86%	7.31%	3.73%	2.10%	5.69%	7.73%
15	JSC NPF Sberbank	9.60%	8.70%	4.64%	8.17%	6.14%	7.44%	2.35%	13.11%
16	JSC NPF Sotsium	10.62%	8.93%	5.61%	8.71%	7.29%	2.29%	4.57%	5.44%
17	JSC NPF Transneft	8.83%	8.39%	3.72%	8.78%	5.02%	2.17%	4.59%	7.12%
18	JSC Khanty-Mansiysk NPF	9.63%	8.16%	4.15%	10.94%	5.42%	1.28%	6.87%	3.97%
19	JSC NPF Future	4.08%	-2.01%	-15.28%	6.87%	4.74%	4.16%	1.79%	11.74%
20	JSC NPF Otkritie	8.23%	-5.26%	-10.80%	10.33%	4.24%	4.14%	4.79%	6.51%
21	JSC NPF Surgutneftegaz	11.61%	8.74%	5.32%	12.12%	6.14%	1.86%	7.32%	5.59%
22	JSC NPF Evolyuciya	10.80%	8.13%	5.12%	10.36%	6.79%	2.52%	5.08%	8.05%

Table 3. Maximum value of the annual return and the intrinsic growth rate of the annualized return calculated by Equations 3 and 4, respectively

No.	NPF	Auxiliary value of annual yield N_p, %	Rate of change of annual yield r, fractions of units.
1	JSC “MNPF AQUILON”	6.47	0.027
2	JSC NPF Almaznaya Osen	6.94	0.011
3	JSC NPF Alliance	15.35	-0.005
4	JSC MNPF BOLSHOI	3.48	-0.002
5	JSC NPF Volga-Capital	4.19	-0.008
6	JSC NPF VTB Pension Fund	6.11	-0.119
7	JSC NPF GAZFOND Pension Accumulation JSC	7.05	-0.142
8	JSC NPF Gefest	3.91	-0.171
9	JSC National NPF	4.52	0.008
10	JSC NPF Doverie	5.63	0.007
11	JSC NPF OPF named after V.V. Livanov	7.07	0.002
12	JSC NPF First Industrial Alliance	5.32	-0.052
13	JSC NPF PERSPECTIVA	5.59	0.001
14	JSC NPF Professionalniy	5.11	-0.022
15	JSC NPF Sberbank	5.94	-0.120
16	JSC NPF Sotsium	5.42	-0.081
17	JSC NPF Transneft	4.83	-0.044
18	JSC Khanty-Mansiysk NPF	3.95	-0.014
19	JSC NPF Future	9.60	-0.137
20	JSC NPF Otkritie	5.99	-0.090
21	JSC NPF Surgutneftegaz	5.35	-0.077
22	JSC NPF Evolyuciya	7.38	-0.126

Table 4. Forecast values of annual return on pension savings of citizens in NPFs of the Russian Federation, averaged over five-year periods

No.	NPF	Ordinal number of the five-year period							
		I	II	III	IV	V	VI	VII	VIII
		Years							
		2024-2028	2029-2033	2034-2038	2039-2043	2044-2048	2049-2053	2054-2058	2059-2063
1	JSC “MNPF AQUILON”	6.43%	6.46%	6.46%	6.46%	6.47%	6.47%	6.47%	6.47%
2	JSC NPF Almaznaya Osen	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%
3	JSC NPF Alliance	2.84%	5.90%	7.76%	9.00%	9.90%	10.57%	11.09%	11.52%
4	JSC MNPF BOLSHOI	5.44%	3.79%	3.66%	3.61%	3.58%	3.56%	3.55%	3.54%
5	JSC NPF Volga-Capital	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%
6	JSC NPF VTB Pension Fund	6.65%	6.25%	6.19%	6.17%	6.16%	6.15%	6.14%	6.14%
7	JSC NPF GAZFOND Pension Accumulation JSC	9.77%	7.56%	7.35%	7.26%	7.21%	7.18%	7.16%	7.15%
8	JSC NPF Gefest	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%
9	JSC National NPF	4.82%	4.60%	4.57%	4.55%	4.55%	4.54%	4.54%	4.54%
10	JSC NPF Doverie	6.58%	5.86%	5.77%	5.73%	5.71%	5.69%	5.68%	5.68%
11	JSC NPF OPF named after V.V. Livanov	8.01%	7.30%	7.20%	7.17%	7.14%	7.13%	7.12%	7.11%
12	JSC NPF First Industrial Alliance	5.34%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%
13	JSC NPF PERSPECTIVA	6.80%	5.86%	5.75%	5.70%	5.68%	5.66%	5.65%	5.64%
14	JSC NPF Professionalniy	6.17%	5.35%	5.25%	5.21%	5.19%	5.17%	5.16%	5.16%
15	JSC NPF Sberbank	9.08%	6.46%	6.24%	6.16%	6.11%	6.08%	6.06%	6.04%
16	JSC NPF Sotsium	5.43%	5.43%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%
17	JSC NPF Transneft	5.79%	5.05%	4.96%	4.92%	4.90%	4.88%	4.88%	4.87%
18	JSC Khanty-Mansiysk NPF	3.96%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%
19	JSC NPF Budushchee	10.67%	9.87%	9.76%	9.72%	9.69%	9.67%	9.66%	9.66%
20	JSC NPF Otkritie	6.24%	6.06%	6.03%	6.02%	6.01%	6.01%	6.01%	6.00%
21	JSC NPF Surgutneftegaz	5.47%	5.38%	5.37%	5.37%	5.36%	5.36%	5.36%	5.36%
22	JSC NPF Evolyuciya	7.72%	7.47%	7.44%	7.42%	7.41%	7.41%	7.40%	7.40%

Step 8. Analyze the results obtained using the Verhulst equation: In this step, the obtained results were verified for adequacy and compliance with common sense and logic. Thus, if the obtained results are adequate, we proceed to the next step of algorithm execution. Otherwise, it is necessary to change the forecasting model parameters, such as the intrinsic growth rate of the annualized interest rate of return (r; see Equations 1 and 4), supporting capacity (K, Equation 1), initial time t_0 and number of forecasting periods, with a subsequent transition to step 5 of this algorithm for a comprehensive system for the optimal distribution of citizens' pension assets in NPFs.

Step 9. Exclude NPFs, investments in which are inappropriate, from further consideration: If, based on the results of applying the Verhulst equation, the expected annual return of an NPF throughout the forecasting period is lower than that of other NPFs, such a fund is excluded from further consideration because investments in such a fund will not lead to the maximization of the objective function (5).

Step 10. Set up the dynamic programming problem on the distribution of investments between NPFs over several years: At this step, it is necessary to formulate a dynamic programming problem on the distribution of investments between NPFs over several years and check the fulfillment of the main assumptions of the model: 1) the total financial result does not depend on investments in other NPFs; 2) the total economic effect is equal to the sum of financial results obtained from investments in selected NPFs; 3) the holder of a personalized pension account acts reasonably; 4) the only determining factor for making a managerial decision on investments in the respective NPF is its expected annual return; 5) investment decisions of the personalized pension account holder are not influenced by the brand, publicity of the NPF and other factors, except for economic ones; 6) during the entire investment period, all NPFs will hold, obtain and (or) renew all necessary permits and licenses of the relevant federal and regional bodies and organizations, on whose decisions the results are based; 5) the investment decisions of the personalized pension account holder are not influenced by the brand, publicity of the NPF and other factors, except for economic ones; 6) during the entire investment period all NPFs will hold, obtain and (or) renew all necessary permits and licenses of the relevant federal and regional bodies and organizations, on whose decisions the results are based; 7) the expected annual return of the considered NPFs for the period of labor activity (40 years, from 2024 to 2063) is determined on the basis of the Verhulst forecasting equation; 8) the constraints are linear, but investments in each NPF are discrete, as they depend on the expected annual return; hence, it is difficult to apply linear programming methods to solve this problem. If these conditions are satisfied, then we proceed to the next step of the algorithm. Otherwise, Step 2 is repeated until the criteria are satisfied.

Step 11. Solve the dynamic programming problem of distributing investments between NPFs in the MS Excel software environment: The problem is solved using the economic and mathematical models (5)–(10) and the built-in "Find a Solution" procedure of the MS Excel software product.

Step 12. Analyze the results obtained: The obtained result should be estimated according to the criterion of maximizing the objective function (5) under limitations (6)–(10). If this condition (the maximum of objective function (5)) is met, it is necessary to analyze the obtained results and develop recommendations for citizens regarding the optimal distribution of their pension savings in NPFs. For managers and administrators of existing and prospective pension schemes and programs, it is necessary to develop directions for their improvement regarding the results obtained during modeling, draw conclusions, and complete the algorithm of a comprehensive system for the optimal distribution of citizens' pension savings in NPFs. Otherwise, return to step 10 of the algorithm.

3- Results

The practical realization of the Verhulst forecasting model was carried out using the example of 22 NPFs listed above. The initial values of the annual percentage rate of return required for modeling were taken from the public data of the Central Bank of the Russian Federation (Bank of Russia) [74] and are presented in Table 2. Thus, it follows from the data shown in Table 2 that in 2008, all NPFs, except JSC NPF Doverie, JSC NPF OPF named after V. V. Livanov, and JSC NPF Professionalniy, had a negative value of the annual percentage rate of return, which may be related to the consequences of the global financial crisis that affected the Russian Federation. At the same time, it should be noted that the JSC NPF OPF named after V. V. Livanov had the highest yield on investments of the funded component of citizens' pensions in 2008 (50.29%, see line 11 of Table 2). Subsequently, it gradually decreased, until it became negative in 2011. We see a higher value of profitability for the whole study period for all 22 NPFs, namely, 55.72% (see line 1 of Table 2) for only the JSC MNPF AQUILON in 2009 after a negative value in 2008. Thus, according to the data presented in Table 2, it can be concluded that from 2008 to 2011, all NPFs considered in Table 2 have high volatility in the values of annual percentage rates of return on deposits of funded pensions of Russian citizens, which is obviously associated with the global financial crisis of 2008 and its consequences. Since 2012, the volatility of annual interest rates has been decreasing for all 22 NPFs and remains low until 2023, both for one NPF in dynamics and for all NPFs in general.

The maximum value of the return on pension savings of citizens in the NPF of the Russian Federation and the intrinsic growth rate of annualized returns are determined using Equations 3 and 4, respectively. The calculation results are listed in Table 3. For example, for JSC MNPF AQUILON, the auxiliary value of annual yield $N_p = 0.07 \cdot \frac{0.0527 \cdot 0.0683 + 0.0683 \cdot 0.0639 - 2 \cdot 0.0527 \cdot 0.0639}{0.0683^2 - 0.0527 \cdot 0.0639} = 0.0647$ or 6.47%, where 0.0527, 0.0683, and 0.0639 represent the values of the annual percentage rate of return on investments of the funded part of the pension funds of Russian citizens in the NPF JSC MNPF AQUILON in 2013, 2018, and 2023, respectively (see the first row of Table 2). The resulting value of 6.47% is reported for JSC MNPF AQUILON in Row 1 of Table 3. This is similar to the other ***auxiliary values of annual yield*** in Table 3.

Let us consider the calculation methodology of ***rate of change of annual yield*** for the example of the same NPF JSC MNPF AQUILON. Thus, according to Equation 4, the value of 0.027, specified in line 1 of Table 3, is equal to $r = \frac{\ln 0.0521 - \ln 0.0398}{2022 - 2012} = 0.027$, where 0.0398 and 0.0521 are the values of the annual percentage rate of return on investments of the funded pensions of Russian citizens in the NPF JSC MNPF AQUILON in 2022 and 2012, respectively (see the first row of Table 2). The obtained value of 0.027 is shown in row 1 of Table 3 for the JSC MNPF AQUILON. This is similar to the other values for the ***rate of change of annual yield*** in Table 3.

Forecast values of the annual return on pension savings of citizens in NPFs of the Russian Federation for the period from 2024 to 2063, calculated according to Verhulst equation (2), are presented in the Appendix I (Table AI). Parameter N_0 represents the annual return on pension savings of citizens in the Russian Federation's NPF initially moment in time, that is, as of 2023 (see the last column of Table 2). As stated above, citizens are recommended to invest their pension savings in NPFs no more than once in five years; otherwise, as shown in the research paper by Kostyrin et al. [1], citizens risk losing investment returns on their invested funds. Table 4 presents the data in the Appendix I (Table AI), grouped and averaged over the corresponding five-year periods.

We solve the dynamic programming problem of the optimal distribution of pension savings of citizens in the 22 NPFs of the Russian Federation under consideration (5)–(10) with the forecast of annual profitability for the period from 2024 to 2063 (see Table 4), using the Verhulst forecasting model (see Table 4) in the MS Excel program environment. For this purpose, we created a working field on an MS Excel spreadsheet with the same dimensions as the initial data (Table 4). The working field (matrix) is filled with arbitrary values such as zero, as shown in Figure 3.

In the cells to the right and bottom of the table, we enter formulas representing the sums of the elements in each row and column of the working matrix, respectively. Thus, the value 0 in cell U6 is the sum of the row elements of the M6-T6 matrix, i.e. U6=M6+N6+O6+P6+Q6+R6+S6+T6, U7=M7+N7+O7+P7+Q7+R7+S7+T7 and so on up to U27. The value of 0 in cell M28 is the sum of the elements of the first column of the working matrix, that is, M28 =M6+M7+M8+...+M27, N28 =N6+N7+N8+...+N27, etc., up to T28. The value 0 in cell U28 is the sum of cells M28-T28 and shows the number of possible transitions from one NPF to another without loss of profitability. The objective function is the sum of the products of the corresponding elements in the two arrays. In the MS Excel program environment, this record appeared as follows: cell C3=SUMPRODUCTION(C6:J27;M6:T27).

As stated above, the problem is to determine the amount of funds allocated for investment in each NPF to maximize the total economic effect. Consequently, the considered dynamic programming problem regarding the optimal distribution of pension savings of citizens in the considered NPFs of the Russian Federation is a maximum search problem. To determine the maximum value of the objective function, we use the "Find a Solution" subprogram built into the MS Excel software product.

	B	C	D	E	F	G	H	I	J	K
1										
2										
3	Target function	0								
4										
5	Name of the NPF	2024-2028	2029-2033	2034-2038	2039-2043	2044-2048	2049-2053	2054-2058	2059-2063	
6	JSC “MNPF AQUILON”	6.43%	6.46%	6.46%	6.46%	6.47%	6.47%	6.47%	6.47%	
7	JSC NPF Almaznaya Osen	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	
8	JSC NPF Alliance	2.84%	5.90%	7.76%	9.00%	9.90%	10.57%	11.09%	11.52%	
9	JSC MNPF BOLSHOI	5.44%	3.79%	3.66%	3.61%	3.58%	3.56%	3.55%	3.54%	
10	JSC NPF Volga-Capital	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	
11	JSC NPF VTB Pension Fund	6.65%	6.25%	6.19%	6.17%	6.16%	6.15%	6.14%	6.14%	
12	JSC NPF GAZFOND Pension Accumulation JSC	9.77%	7.56%	7.35%	7.26%	7.21%	7.18%	7.16%	7.15%	
13	JSC NPF Gefest	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	
14	JSC National NPF	4.82%	4.60%	4.57%	4.55%	4.55%	4.54%	4.54%	4.54%	
15	JSC NPF Doverie	6.58%	5.86%	5.77%	5.73%	5.71%	5.69%	5.68%	5.68%	
16	JSC NPF OPF named after V.V. Livanov	8.01%	7.30%	7.20%	7.17%	7.14%	7.13%	7.12%	7.11%	
17	JSC NPF First Industrial Alliance	5.34%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	
18	JSC NPF PERSPECTIVA	6.80%	5.86%	5.75%	5.70%	5.68%	5.66%	5.65%	5.64%	
19	JSC NPF Professionalniy	6.17%	5.35%	5.25%	5.21%	5.19%	5.17%	5.16%	5.16%	
20	JSC NPF Sberbank	9.08%	6.46%	6.24%	6.16%	6.11%	6.08%	6.06%	6.04%	
21	JSC NPF Sotsium	5.43%	5.43%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%	
22	JSC NPF Transneft	5.79%	5.05%	4.96%	4.92%	4.90%	4.88%	4.88%	4.87%	
23	JSC Khanty-Mansiysk NPF	3.96%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	
24	JSC NPF Budushchee	10.67%	9.87%	9.76%	9.72%	9.69%	9.67%	9.66%	9.66%	
25	JSC NPF Otkritie	6.24%	6.06%	6.03%	6.02%	6.01%	6.01%	6.01%	6.00%	
26	JSC NPF Surgutneftegaz	5.47%	5.38%	5.37%	5.37%	5.36%	5.36%	5.36%	5.36%	
27	JSC NPF Evolyuciya	7.72%	7.47%	7.44%	7.42%	7.41%	7.41%	7.40%	7.40%	
28										

L	M	N	O	P	Q	R	S	T	U
Name of the NPF	2024-2028	2029-2033	2034-2038	2039-2043	2044-2048	2049-2053	2054-2058	2059-2063	
JSC “MNPF AQUILON”	0	0	0	0	0	0	0	0	0
JSC NPF Almaznaya Osen	0	0	0	0	0	0	0	0	0
JSC NPF Alliance	0	0	0	0	0	0	0	0	0
JSC MNPF BOLSHOI	0	0	0	0	0	0	0	0	0
JSC NPF Volga-Capital	0	0	0	0	0	0	0	0	0
JSC NPF VTB Pension Fund	0	0	0	0	0	0	0	0	0
JSC NPF GAZFOND Pension Accumulation JSC	0	0	0	0	0	0	0	0	0
JSC NPF Gefest	0	0	0	0	0	0	0	0	0
JSC National NPF	0	0	0	0	0	0	0	0	0
JSC NPF Doverie	0	0	0	0	0	0	0	0	0
JSC NPF OPF named after V.V. Livanov	0	0	0	0	0	0	0	0	0
JSC NPF First Industrial Alliance	0	0	0	0	0	0	0	0	0
JSC NPF PERSPECTIVA	0	0	0	0	0	0	0	0	0
JSC NPF Professionalniy	0	0	0	0	0	0	0	0	0
JSC NPF Sberbank	0	0	0	0	0	0	0	0	0
JSC NPF Sotsium	0	0	0	0	0	0	0	0	0
JSC NPF Transneft	0	0	0	0	0	0	0	0	0
JSC Khanty-Mansiysk NPF	0	0	0	0	0	0	0	0	0
JSC NPF Budushchee	0	0	0	0	0	0	0	0	0
JSC NPF Otkritie	0	0	0	0	0	0	0	0	0
JSC NPF Surgutneftegaz	0	0	0	0	0	0	0	0	0
JSC NPF Evolyuciya	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0

Figure 3. Fragment of MS Excel sheet with initial values for solving problem (5)-(10)

In the corresponding field of the dialog box, the target cell $C$3 is set equal to the maximum value (maximization problem). Cells $M$6:$T$27 are the changeable cells in the considered dynamic programming problem regarding the optimal distribution of pension savings of citizens in the NPF of the Russian Federation, so we specify them in the corresponding field of the dialog box. In the “Restrictions” field, we specify the existing restrictions on the cells to be changed. In the task under consideration, the following constraints exist.

- Only one NPF can be selected in each of the five-year time periods, which is specified by the following restriction: $M$28:$T$38=1;
- The values of the cells to be changed can only be binary (binary) 0 or 1;
- U28 is the sum of the number of managerial decisions made (choices made by citizens) in which the NPF should invest. In total, there should be as many choices as time periods, that is, 8.

After creating the above constraints, click “Find a solution.” The solution of the dynamic programming problem on the optimal distribution of pension savings for citizens in the NPF of the Russian Federation in the MS Excel program environment is shown in Figure 4.

Target function 5202,01%

№	Name of the NPF	2024-2028	2029-2033	2034-2038	2039-2043	2044-2048	2049-2053	2054-2058	2059-2063
1	JSC "MNPF AQUILON"	6.43%	6.46%	6.46%	6.46%	6.47%	6.47%	6.47%	6.47%
2	JSC NPF Almaznaya Osen	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%
3	JSC NPF Alliance	2.84%	5.90%	7.76%	9.00%	9.90%	10.57%	11.09%	11.52%
4	JSC MNPF BOLSHOI	5.44%	3.79%	3.66%	3.61%	3.58%	3.56%	3.55%	3.54%
5	JSC NPF Volga-Capital	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%
6	JSC NPF VTB Pension Fund	6.65%	6.25%	6.19%	6.17%	6.16%	6.15%	6.14%	6.14%
7	JSC NPF GAZFOND Pension Accumulation JSC	9.77%	7.56%	7.35%	7.26%	7.21%	7.18%	7.16%	7.15%
8	JSC NPF Gefest	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%
9	JSC National NPF	4.82%	4.60%	4.57%	4.55%	4.55%	4.54%	4.54%	4.54%
10	JSC NPF Doverie	6.58%	5.86%	5.77%	5.73%	5.71%	5.69%	5.68%	5.68%
11	JSC NPF OPF named after V.V. Livanov	8.01%	7.30%	7.20%	7.17%	7.14%	7.13%	7.12%	7.11%
12	JSC NPF First Industrial Alliance	5.34%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%
13	JSC NPF PERSPECTIVA	6.80%	5.86%	5.75%	5.70%	5.68%	5.66%	5.65%	5.64%
14	JSC NPF Professionalniy	6.17%	5.35%	5.25%	5.21%	5.19%	5.17%	5.16%	5.16%
15	JSC NPF Sberbank	9.08%	6.46%	6.24%	6.16%	6.11%	6.08%	6.06%	6.04%
16	JSC NPF Sotsium	5.43%	5.43%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%
17	JSC NPF Transneft	5.79%	5.05%	4.96%	4.92%	4.90%	4.88%	4.88%	4.87%
18	JSC Khanty-Mansiysk NPF	3.96%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%
19	JSC NPF Budushchee	10.67%	9.87%	9.76%	9.72%	9.69%	9.67%	9.66%	9.66%
20	JSC NPF Otkritie	6.24%	6.06%	6.03%	6.02%	6.01%	6.01%	6.01%	6.00%
21	JSC NPF Surgutneftegaz	5.47%	5.38%	5.37%	5.37%	5.36%	5.36%	5.36%	5.36%
22	JSC NPF Evolyuciya	7.72%	7.47%	7.44%	7.42%	7.41%	7.41%	7.40%	7.40%

Name of the NPF	2024-2028	2029-2033	2034-2038	2039-2043	2044-2048	2049-2053	2054-2058	2059-2063	
JSC "MNPF AQUILON"	0	0	0	0	0	0	0	0	0
JSC NPF Almaznaya Osen	0	0	0	0	0	0	0	0	0
JSC NPF Alliance	0	0	0	0	1	1	1	1	4
JSC MNPF BOLSHOI	0	0	0	0	0	0	0	0	0
JSC NPF Volga-Capital	0	0	0	0	0	0	0	0	0
JSC NPF VTB Pension Fund	0	0	0	0	0	0	0	0	0
JSC NPF GAZFOND Pension Accumulation JSC	0	0	0	0	0	0	0	0	0
JSC NPF Gefest	0	0	0	0	0	0	0	0	0
JSC National NPF	0	0	0	0	0	0	0	0	0
JSC NPF Doverie	0	0	0	0	0	0	0	0	0
JSC NPF OPF named after V.V. Livanov	0	0	0	0	0	0	0	0	0
JSC NPF First Industrial Alliance	0	0	0	0	0	0	0	0	0
JSC NPF PERSPECTIVA	0	0	0	0	0	0	0	0	0
JSC NPF Professionalniy	0	0	0	0	0	0	0	0	0
JSC NPF Sberbank	0	0	0	0	0	0	0	0	0
JSC NPF Sotsium	0	0	0	0	0	0	0	0	0
JSC NPF Transneft	0	0	0	0	0	0	0	0	0
JSC Khanty-Mansiysk NPF	0	0	0	0	0	0	0	0	0
JSC NPF Budushchee	1	1	1	1	0	0	0	0	4
JSC NPF Otkritie	0	0	0	0	0	0	0	0	0
JSC NPF Surgutneftegaz	0	0	0	0	0	0	0	0	0
JSC NPF Evolyuciya	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	8

Figure 4. Solution of dynamic programming problem in MS Excel software environment

As can be seen from the results presented in Figure 4, the maximum return on investment of citizens' pension funds will be provided in the period from 2024 to 2043, funds are invested in NPF JSC "Future," and from 2044 to 2063 the funds are invested in NPF JSC "Alliance" (units in the corresponding cells of the table in Figure 4). The total return on pension savings for the entire investment period (40 years) amounts to 5202%, or more than 52 times the amount of the initial investments (see the value of the objective function in cell C3 in Figure 4).

4- Discussion

- On the topic of this study, 80 scientific articles by domestic and foreign researchers and specialists were analyzed, and divided into the following seven groups: 1. autoregressive moving-average model (ARIMA) (20 publications); 2. AutoRegressive Conditional Heteroscedasticity, ARCH (18 publications); 3. Vector AutoRegression, VAR (seven publications), 4. Machine learning, ML (14 publications), and 5. neural networks, NN (12 publications), and 6. deep learning, DL (nine publications), and 7. Logistic equation, EL (eight publications). A comparative analysis of the models and methods used for forecasting, presented in Figure 1 and Table 1, shows that there are currently no studies devoted to the development of easy-to-learn and easy-to-apply practical models that do not require special training and high qualifications of performers, accurate and high-quality methods, or models for forecasting the annual interest rate of return. Simultaneously, the use of the Verhulst equation for these purposes, which has been actively applied in the implementation of demographic forecasts, deserves special interest.

- An innovative approach to the optimal distribution of citizens' pension savings to non-state pension funds was developed. It allows automating the process of making managerial decisions on the selection of NPFs and the optimal distribution of the funded part of citizens' pension funds to maximize the total return on pension savings of Russian citizens for the entire investment period (40 years). The practical implementation of the developed approach using open data from the Social Fund of Russia on annual interest rates of NPF profitability was also conducted [86].
- This study analyzes the annual return on investment of citizens' pension assets in non-state pension funds of the Russian Federation (22 key NPFs are considered) for the period from 2008 to 2023 and concludes that in the Russian Federation there has been long-term economic and legal tension in the distribution of citizens' pension savings. However, at the same time, citizens have the opportunity to choose an organization that manages the funded component of their pensions, which makes it possible to optimally distribute assets to obtain maximum income. As shown in the data presented in the last column of Table 2, the highest return on citizens' pension investments in annual terms as of 2023 was noted for the joint-stock company Non-State Pension Fund Sberbank (13.11%), and the lowest for the non-state pension fund Alliance (1.13%). It should also be noted that most non-state pension funds had negative returns in 2008, which is likely associated with the consequences of the global financial crisis of 2008. Thus, according to the data in Table 2, the largest loss was noted for Gefest NPF, namely -71.26%, but there are exceptions to this rule, for example, V. V. Livanov OPF NPF with an annualized gain of 50.29%.
- Comprehensive economic and mathematical models (1)–(10) have been developed, which include an interconnected prognostic model based on the Verhulst and Bellman equations, an operation flowchart of the algorithm for the practical implementation of the developed model (see Figure 1), and MS Excel-based software, which provides the foundation for creating a scientifically sound, complete, and consistent decision-making system for managers and administrators of existing and prospective pension schemes and programs.
- This study proposes the use of the Verhulst equation as a tool for forecasting the value of the interest rate for the annual rate of return of non-state pension funds. The use of this model makes it possible to forecast the annual rate of return accurately using a minimum number of resources.
- This study proposes an approach for solving the problem of investment allocation in NPFs. This approach allowed us to determine the NPF for which it is most profitable to invest in pension contributions. The use of the author's tools based on the method of dynamic programming, built-in procedures of MS Excel software, allows for the automation of the process of making managerial decisions regarding the choice of NPF.
- The forecast for the period from 2024 to 2063 using the Verhulst forecasting model developed in the article showed that the highest value of the expected return in 2063, namely 11.66% in annual terms, should be expected from JSC NPF "Alliance", and the minimum (3.54% per year) – from JSC MNPF "BOLSHOY".

5- Conclusions and Policy Recommendations

5-1- Findings of the Study

The solution of the problem of dynamic programming on the optimal distribution of pension savings of citizens in NPF has shown that the maximum return on investment of pension funds of citizens will be under the condition that in the period from 2024 to 2043 the funds are invested in JSC "NPF Future, and from 2044 to 2063 the funds are invested in JSC NPF Alliance. The total return on pension savings for the entire investment period (40 years) amounts to 5202%, or more than 52 times the initial investments.

5-2- Strengths and Limitations

The advantages of the methodology proposed in this study include the combined use of dynamic programming and predictive Verhulst equation methods, which facilitate the determination of the optimal strategy for investing citizens' assets in non-state pension funds to maximize the total return on investment over the period of working activity. In addition, the solution to the problem in the widely used MS Excel software product shown in this study makes it possible to automate the process of making managerial decisions and calculate various modeling options depending on the initial parameters in real time. The profitability of non-state pension funds, average duration of working activity, volume of citizens' financial resources available for distribution between non-state pension funds, and other modeling parameters are the parameters used in this study's framework. Another strength of this study is the solution of the dynamic programming problem as a transport problem, which allows the use of simple and widespread linear programming methods to solve such problems, such as the simplex method. Notably, the application of the Verhulst equation, primarily developed and employed for demographic forecasting, to predict the profitability of non-state pension funds significantly expands the variety of economic and mathematical modeling tools and financial technologies available to address pressing issues in managing citizens' financial resources for optimal investment. This approach may hold practical value for a diverse range of investors and managers in the securities and financial asset markets.

Model limitations: 1) the total financial result does not depend on investments in other NPFs; 2) the total economic effect is equal to the sum of financial results obtained from investments in selected NPFs; 3) the owner of a personalized pension account acts reasonably; 4) the only determining factor for making a managerial decision on investments in the respective NPF is its expected annual return; 5) investment decisions of the personalized pension account holder are not influenced by the brand, publicity of the NPF and other factors, except for economic ones; 6) during the entire investment period, all NPFs will hold, obtain and (or) renew all necessary permits and licenses of the relevant federal and regional bodies and organizations, on whose decisions the results are based; 5) the investment decisions of the personalized pension account holder are not influenced by the brand, publicity of the NPF and other factors, except for economic ones; 6) during the entire investment period all NPFs will hold, obtain and (or) renew all necessary permits and licenses of the relevant federal and regional bodies and organizations, on whose decisions the results are based; 7) the expected annual return of the considered NPFs for the period of labor activity (40 years, from 2024 to 2063) is determined on the basis of the Verhulst forecasting equation; 8) the constraints are linear, but investments in each NPF are discrete, as they depend on the expected annual return, so it is difficult to apply linear programming methods to solve this problem; 9) another model limitation is that, as shown in Figure 4, the study models a situation where all pension assets are invested in one non-state pension fund for five years, while the option of portfolio investment of pension assets in different shares in several non-state pension funds with different gain and risk levels is of practical interest, which will allow diversification of investments.

5-3- Comparison with the Results of Previous Studies and Scientific Accumulation of Knowledge

The innovative approach to the optimal allocation of pension savings of citizens in the NPF proposed in this study differs from other known models in that, unlike, for example, the existing works of specialists in the field of building predictive models [6, 7, 23-25, 44, 49, 52], artificial intelligence [48, 50, 62, 63, 72, 74], and actuarial calculations in the field of state pension insurance [87-92]. This study proposes a comprehensive management decision support system that includes a predictive model based on the Verhulst equation, a dynamic programming model, Bellman equations, a block diagram of the algorithm for the practical implementation of the developed model, and MS Excel-based software that provides a basis for creating a scientifically sound, complete, and consistent decision-making system for managers and administrators of existing and prospective pension schemes and programs.

5-4- Recommendations for Future Research

Further research areas may include:

- Improving the accuracy and predictive value of the Verhulst equation developed to solve financial managerial problems in other areas of research, such as average life expectancy, to link forecasts of investment gains in non-state pension funds with the average pension size for the post-retirement survival period (the period from retirement to death).
- Analyzing existing forecasting models and their capabilities for assessing the economic efficiency of investments in non-state pension funds using elements of machine learning, artificial intelligence, autoregressive models, and polynomial dependence models to avoid errors at the stage of model construction and loss of income for citizens at the stage of practical implementation of models with undetected errors and inaccuracies that may subsequently lead to the effect of error accumulation, which will significantly reduce the practical value of such models and their forecasts.
- Introducing feedback into the forecasting model allows the decision-maker to adjust the model parameters in a timely manner and avoid unjustified decisions and serious consequences from their implementation.
- Extending the solution of the dynamic programming problem to the state pension fund and considering and introducing the successful experience of other countries into the model, for example, personalized pension accounts, have proven themselves well in Singapore [93, 94].
- Increasing the sampling frequency in the solution of the dynamic programming problem from five years, as conditioned by the analysis of the legislation of the Russian Federation, to one year will allow for more accurate solutions and provide citizens with the freedom to make managerial decisions on investing their pension assets.
- Analyzing the risks of investing citizens' pension assets in non-state pension funds and the possibilities of investment diversification by investing different shares of assets in various non-state pension funds with different gains and risk levels than the entire amount in non-state pension funds, as shown in Figure 4.

- Developing an economic and mathematical model for managing citizens' pension assets using personalized pension accounts based on nonlinear programming methods and econometric models that are more adequate for the tasks at hand, including a wide range of financial technologies and modeling parameters, will allow for the development of not only mechanisms for investing citizens' pension savings but also modeling properties of pension assets such as inheritance transfers, regulation of average pension amounts for post-retirement survival periods, and/or the duration of pension benefits. Additionally, it allows for more extensive asset utilization, including the transfer of funds from individual pension accounts to healthcare savings plans, mortgage initiatives for property acquisition, and other purposes [95].

The theoretical significance of this study lies in the development of unique methods for the combined use of Verhulst equation-based forecasting models with dynamic programming methods to determine optimal investments in various non-state pension funds during the period of employment. This significantly expands the arsenal of economic and technical means for solving problems of managing citizens' financial resources and their savings, which is a pressing national economic problem.

The practical importance lies in the practical application of dynamic programming methods as a transport problem for determining the most profitable investments of citizens' pension assets and solving the dynamic programming problem in the widely used MS Excel software product, which allows the automation of the process of managerial decision-making and makes the technology developed by the authors within the framework of this study accessible and widespread among managers in pension provision, citizens, and other areas of financial resource management.

Policy Recommendations: The methodology developed by the authors can effectively use governmental administrative bodies to simulate the profitability of pension schemes for diverse citizen groups by considering age, gender, income, and life expectancy. This method enables the evaluation of both public and private pension fund stability and equilibrium as well as the development and verification of new and existing pension initiatives. This will increase the reliability, accuracy, and validity of decisions taken and will provide an opportunity to consider and analyze possible options for modeling and scenarios for the socio-economic development of regions, countries, and even integration associations. Furthermore, this research provides a robust and efficient set of tools, procedures, and software for implementing social security measures, such as personalized retirement savings accounts, individual health savings accounts, and other forms of citizen social support, ensuring the fulfillment of governmental obligations.

Managerial Implications: For managers in the field of financial resource management, this study may be interesting and useful as a tool for managing finances in various areas of activity, for example, in securities markets, analyzing and managing the financial results of enterprises, solving problems of optimal investment distribution between several enterprises with different profitability, and developing loyalty programs for enterprises' employees and clients. Thus, of particular interest are voluntary pension and medical insurance schemes for employees using progressive labor incentive systems, which can be based on the software and algorithms developed in this study and Verhulst equation-based forecasting models

6- Declarations

6-1- Author Contributions

Conceptualization, E.K. and S.D.; methodology, E.K.; software, S.D.; validation, E.K. and S.D.; formal analysis, S.D.; investigation, E.K.; resources, E.K. and S.D.; data curation, E.K.; writing—original draft preparation, E.K. and S.D.; writing—review and editing, E.K.; visualization, S.D.; supervision, E.K.; project administration, E.K.; funding acquisition, S.D. All authors have read and agreed to the published version of the manuscript.

6-2- Data Availability Statement

The data presented in this study are available in the article.

6-3- Funding

The authors received no financial support for the research, authorship, and/or publication of this article.

6-4- Institutional Review Board Statement

Not applicable.

6-5- Informed Consent Statement

Not applicable.

6-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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Appendix I

Table AI. Forecast values of annual return on pension savings of citizens in NPFs of the Russian Federation for the period from 2024 to 2063, calculated using the Verhulst forecasting equation (2024-2033)

No.	NPF	Year									
		2024	20325	2026	2027	2028	2029	2030	2031	2032	2033
1	JSC “MNPF AQUILON”	6.39%	6.43%	6.44%	6.45%	6.45%	6.46%	6.46%	6.46%	6.46%	6.46%
2	JSC NPF Almaznaya Osen	6.93%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%
3	JSC NPF Alliance	1.12%	2.10%	2.94%	3.69%	4.35%	4.94%	5.47%	5.95%	6.38%	6.78%
4	JSC MNPF BOLSHOI	9.57%	5.10%	4.42%	4.14%	3.99%	3.89%	3.83%	3.78%	3.74%	3.72%
5	JSC NPF Volga-Capital	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%
6	JSC NPF VTB Pension Fund	7.37%	6.68%	6.48%	6.38%	6.33%	6.29%	6.26%	6.24%	6.23%	6.22%
7	JSC NPF GAZFOND Pension Accumulation	14.77%	9.54%	8.54%	8.11%	7.87%	7.72%	7.62%	7.54%	7.48%	7.44%
8	JSC NPF Gefest	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%
9	JSC National NPF	5.22%	4.85%	4.73%	4.68%	4.64%	4.62%	4.61%	4.60%	4.59%	4.58%
10	JSC NPF Doverie	7.98%	6.60%	6.25%	6.08%	5.99%	5.92%	5.88%	5.85%	5.82%	5.80%
11	JSC NPF OPF named after V.V. Livanov	9.35%	8.05%	7.69%	7.53%	7.43%	7.37%	7.32%	7.29%	7.27%	7.24%
12	JSC NPF First Industrial Alliance	5.36%	5.34%	5.33%	5.33%	5.33%	5.33%	5.32%	5.32%	5.32%	5.32%
13	JSC NPF PERSPECTIVA	8.70%	6.80%	6.34%	6.14%	6.02%	5.94%	5.89%	5.85%	5.82%	5.79%
14	JSC NPF Professionalniy	7.82%	6.18%	5.78%	5.59%	5.49%	5.42%	5.38%	5.34%	5.31%	5.29%
15	JSC NPF Sberbank	15.50%	8.59%	7.48%	7.03%	6.78%	6.63%	6.52%	6.44%	6.38%	6.34%
16	JSC NPF Sotsium	5.44%	5.43%	5.43%	5.43%	5.43%	5.43%	5.43%	5.43%	5.43%	5.43%
17	JSC NPF Transneft	7.28%	5.80%	5.44%	5.27%	5.17%	5.11%	5.07%	5.04%	5.01%	4.99%
18	JSC Khanty-Mansiysk NPF	3.97%	3.96%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%
19	JSC NPF Future	12.14%	10.72%	10.32%	10.13%	10.02%	9.95%	9.90%	9.86%	9.83%	9.81%
20	JSC NPF Otkritie	6.56%	6.26%	6.17%	6.12%	6.10%	6.08%	6.07%	6.06%	6.05%	6.04%
21	JSC NPF Surgutneftegaz	5.61%	5.48%	5.44%	5.42%	5.40%	5.39%	5.39%	5.38%	5.38%	5.38%
22	JSC NPF Evolyuciya	8.15%	7.75%	7.62%	7.56%	7.52%	7.50%	7.48%	7.47%	7.46%	7.45%

Table AI (continued). Forecast values of annual return on pension savings of citizens in NPFs of the Russian Federation for the period from 2024 to 2063, calculated using the Verhulst forecasting equation (2034-2043)

No.	NPF	Year									
		2034	2035	2036	2037	2038	2039	2040	2041	2042	2043
1	JSC “MNPF AQUILON”	6.46%	6.46%	6.46%	6.46%	6.46%	6.46%	6.46%	6.46%	6.46%	6.46%
2	JSC NPF Almaznaya Osen	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%	6.94%
3	JSC NPF Alliance	7.14%	7.47%	7.78%	8.07%	8.33%	8.57%	8.80%	9.02%	9.22%	9.40%
4	JSC MNPF BOLSHOI	3.69%	3.67%	3.66%	3.65%	3.63%	3.62%	3.62%	3.61%	3.60%	3.59%
5	JSC NPF Volga-Capital	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%	4.19%
6	JSC NPF VTB Pension Fund	6.21%	6.20%	6.19%	6.19%	6.18%	6.18%	6.17%	6.17%	6.17%	6.16%
7	JSC NPF GAZFOND Pension Accumulation	7.40%	7.37%	7.34%	7.32%	7.30%	7.29%	7.27%	7.26%	7.25%	7.24%
8	JSC NPF Gefest	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%	3.91%
9	JSC National NPF	4.58%	4.57%	4.57%	4.56%	4.56%	4.56%	4.56%	4.55%	4.55%	4.55%
10	JSC NPF Doverie	5.79%	5.78%	5.76%	5.75%	5.75%	5.74%	5.73%	5.73%	5.72%	5.72%
11	JSC NPF OPF named after V.V. Livanov	7.23%	7.21%	7.20%	7.19%	7.18%	7.18%	7.17%	7.17%	7.16%	7.16%
12	JSC NPF First Industrial Alliance	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%	5.32%
13	JSC NPF PERSPECTIVA	5.77%	5.76%	5.74%	5.73%	5.72%	5.71%	5.71%	5.70%	5.69%	5.69%
14	JSC NPF Professionalniy	5.28%	5.26%	5.25%	5.24%	5.23%	5.22%	5.22%	5.21%	5.20%	5.20%
15	JSC NPF Sberbank	6.30%	6.27%	6.24%	6.22%	6.20%	6.18%	6.17%	6.16%	6.14%	6.13%
16	JSC NPF Sotsium	5.43%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%	5.42%
17	JSC NPF Transneft	4.98%	4.97%	4.95%	4.94%	4.94%	4.93%	4.92%	4.92%	4.91%	4.91%
18	JSC Khanty-Mansiysk NPF	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%	3.95%
19	JSC NPF Future	9.79%	9.77%	9.76%	9.75%	9.74%	9.73%	9.72%	9.71%	9.71%	9.70%
20	JSC NPF Otkritie	6.04%	6.03%	6.03%	6.03%	6.03%	6.02%	6.02%	6.02%	6.02%	6.02%
21	JSC NPF Surgutneftegaz	5.38%	5.37%	5.37%	5.37%	5.37%	5.37%	5.37%	5.37%	5.37%	5.37%
22	JSC NPF Evolyuciya	7.45%	7.44%	7.44%	7.43%	7.43%	7.43%	7.42%	7.42%	7.42%	7.42%