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# A New Approach to the Use of Non-Primitive Variables in the Mechanics of Continuous Media

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**Abstract**

The problem of an approximate solution to hydrodynamic problems is the consideration of pressure. To exclude it from the equations, the transition to "non-primitive variables" (vortex and velocity vector divergence) is made. In this case, there are difficulties in the algorithmization of new equations for solving the inverse problem of hydrodynamics and a lot of internal iterative calculations. The object of this study includes equations in "non-primitive" variables. The research methods are based on the transformation without simplifications and assumptions of hydrodynamic equations into a form containing "non-primitive" variables and the demonstration of the possibilities of solving the equations. The GAMS programming language was used for approximate solutions for the first time. The aim of this paper is to demonstrate the possibility of solving the full equations in "non-primitive" variables for various conditions. The results showed the possibility of considering the compressibility of the medium when solving the inverse problem of hydrodynamics; the identity of solutions of the proposed system of equations and equations using the potential; and the possibility of using optimizing programming languages for hydrodynamics problems. The scientific novelty of this research consists of solving the full equations of hydrodynamics with the use of "non-primitive" variables but without the use of the current function.

**Keywords:**
Continuous Medium;
Non-Primitive Variable;
Vortex;
Motion Vector Potential;
Compressibility of the Medium.

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## 1- Introduction

To solve problems regarding the motion of a continuous medium, such as water and gas, familiar and understandable variables such as pressure and velocity are usually used. One of the problems in developing algorithms for solving fluid dynamics problems in "primitive" variables is the calculation of pressure [1–3]. Many hydrodynamicists [2–5] point out the problem that arises in the algorithmization of the equations of motion. Hydrodynamicists most often relate this problem specifically to pressure calculations [6–8]. In the equations of fluid dynamics, the combination of summands containing pressure and acceleration forms a wave system. This wave system describes the motion of waves on the surface of liquid and sound waves under the compressibility of the liquid or gas. The velocities of propagation of waves on the surface of a liquid or sound waves in a liquid or gas are so large that calculation is only possible with very small-time steps. Small time steps favor the manifestation of scheme viscosity in approximate solutions [9–12]. Without artificial measures, simplifications in the hydrodynamic equations that increase the time step, the schematic viscosity will smooth the approximate solutions to such an extent that they become uninformative and of no interest to either researchers or practicing engineers.

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There are several ways of dropping a number of terms in the equations of fluid dynamics so that they no longer describe the propagation of high-speed waves. To eliminate sound and shock waves, it is sufficient to assume the incompressibility of a liquid or gas and to assume that the hydrodynamic pressure is equal to the hydrostatic pressure [12-14]. In almost all hydrodynamic models using "primitive" variables (velocity and pressure), it is assumed that the moving gas or liquid flow is incompressible, the vertical dimensions of the computational domain are small compared to the horizontal ones, and the hydrodynamic pressure is always exactly equal to the hydrostatic pressure [2, 15, 16]. However, even in this case, it is necessary to solve the problem of the occurrence of waves on the free surface of the water. Their speed is sometimes orders of magnitude higher than the physical displacement of water or gas masses. This means that the schematic viscosity manifests itself in approximate calculations and loses its informative value.

This problem is so difficult to solve that successful attempts have been made to write hydromechanical equations of motion in such a way that they do not contain the explicit presence of pressure as an unknown quantity [17-19]. The method for removing pressure in explicit notation from the equations of hydrodynamics is the method of transitioning to "non-primitive" variables. Pressure will undoubtedly continue to affect the motion of a continuous medium. However, when solving hydrodynamic problems, it does not have to be determined at each iterative step of the calculation. The pressure can be determined separately from the equation written specifically for pressure determination [18, 19]. The essence of the method of transition to "non-primitive" variables is as follows: each term of the equation of motion of a continuous medium is a vector, and accordingly, the "vortex" operation can be applied to each term of the equation of motion [18-20].

The following can be seen here. The pressure in the equations of hydrodynamics is written in the differential operator grad (). The gradient vortex of any field is identically zero. Thus, the equations of motion in hydrodynamics, after applying the vortex operation, will no longer explicitly contain the value of pressure. Of course, pressure will still be involved in the organization of motion, but it will be implicit. Its magnitude can, if necessary, be calculated using a separate equation [19, 20]. The "primitive" variable velocity vector becomes the "non-primitive" variable velocity vector vortex. The main part of this paper will show all identical transformations of "primitive" variables into "non-primitive" variables. The equation of momentum transfer in the system of hydrodynamic equations is transformed into the equation for the transfer of the velocity vector vortex along the velocity vector field. This means that in the approximate solution of the velocity vector vortex transport equation and in each iterative step, it is necessary to know the velocity field according to which the vortices are transported in space. To determine this velocity field, it is necessary to solve the inverse hydrodynamic problem.

The inverse hydrodynamic problem is a problem in which the determination of the "primitive" variable of the velocity vector is carried out on the basis of the knowledge of two "non-primitive" variables: the vorticity of the velocity vector and the divergence of the velocity vector. The problem of algorithmizing the initial equations with "primitive" variables has been transformed into the problem of solving the inverse hydrodynamic problem. The literature review below the text shows the main methods and recent advances in solving the inverse hydrodynamic problem. At present, there are algorithms for solving the inverse problem of hydrodynamics that give acceptable approximate solutions for the calculation of the steady-state motion of incompressible media and, as a rule, for two-dimensional problems. The studies presented in this paper show that equations written with "non-primitive" variables can be solved for compressible media and transient processes. In the approach proposed by the authors and tested in numerical experiments, there are no restrictions on the dimensionality of the solution domain.

Only the speed of the computational technique is a factor that limits the application of methods and approaches to the solution of hydrodynamic problems in three-dimensional domains. Obviously, further development of computer technology will soon remove this technical limitation for good. There are no theoretical limitations to the application of the methods described below to the solution of equations in "non-primitive" variables in three-dimensional spaces.

## 2- Literature Review

In methods for obtaining approximate solutions to hydrodynamic equations, the question of solving evolutionary equations has been very well studied [12, 16, 19]. Evolutionary equations are equations in which there is a summand related to the rate of change of the variable of interest over time. The equations of motion are evolutionary in terms of the components of the velocity vector. The conservation equations are evolutionary with respect to the density of the moving medium or some impurity carried by the medium. First of all, it is connected with the manifestation of the laws of conservation and transfer in space of the momentum of motion and the mass of matter. Even when solving evolutionary equations, which are well amenable to algorithmization, there are computational problems, as mentioned in [3, 21, 22]. These include problems related to stability, accuracy of approximation, and exact compliance with conservation laws. A description of these problems is best given in [3]. Much research has been conducted to find computational schemes that do not suffer from these problems [3, 10, 14].

Unfortunately, the question of the annihilation of all problems is still open, and not all solutions have been found. For example, in some velocity fields, the calculation of the momentum transfer of motion or matter no longer satisfies the conservation laws when known calculation schemes are used [10, 23, 24]. More recently, a scheme for solving evolution equations has been developed that can obey the conservation laws of momentum and mass exactly under any varying velocity field [3, 24]. This scheme has been used and is described in detail in this paper. The use of this scheme is an important condition for obtaining stable, approximate solutions. The point is that the velocity vortex also possesses the property of conservatism as well as the momentum of motion. However, the problem of the absence of the evolution equation for hydrodynamic pressure is somewhat different.

Practical methods for solving the problem of the absence of an evolution equation for pressure can be found in any study devoted to solving hydrodynamic problems with "primitive" variables. Among them there are shifted grids, complicated schemes of different accuracy, "cuts" of the vertical momentum conservation equation to hydrostatics, and the assumption of incompressibility for separate equations [11, 25]. The work on finding an efficient solution to equations in "primitive" variables continues, as does the work on finding an efficient solution to equations in "non-primitive" variables.

The direction of solving the equations of motion in "non-primitive" variables includes the problem of efficiently solving the Poisson equation at each iterative step of vortex transport. In the general case and in three-dimensional space, this involves solving three Poisson equations for each of the three spatial components of the velocity field vortex vector and, at each time step, for the vortex transport equation. This task was time-consuming because of conventional programming languages and the lack of efficient solution algorithms. Therefore, a method has been developed to transition from solving the full inverse hydrodynamic problem, which involves solving three Poisson equations in three-dimensional space, to a planar version of the inverse hydrodynamic problem where only one Poisson equation needs to be solved [8, 19, 26]. This transition becomes possible when a special new variable, called the current function, is introduced into the calculation [16, 18, 20]. The condition that allows the introduction of a new variable, the current function, is the assumption that the medium is incompressible. In addition, all solved problems can only be two-dimensional since the current function is not defined in space. The approach of solving the inverse hydrodynamic problem using the current function is implemented in the algorithms of all researchers working with "non-primitive variables" in hydrodynamic problems [20, 27, 28].

In recent research, mesh [28] and meshless methods [27, 29, 30] have primarily been used to solve Poisson's equations for two-dimensional space. Matsumoto and Hanawa [28] proposed solving Poisson's equations using a difference equation and nested grid methods. According to these authors, the computational burden of iterations is proportional to the total number of cells in the nested grid, and large nested grid sizes are required to obtain stable solutions. Meanwhile, the greatest advantage of Kuhnert's [27] meshless method for solving Poisson's equations for incompressible media is that it can be applied to numerically solve elliptic equations within complex geometry where the mesh is poorly constructed.

According to Ma et al. [30], the chief methods used to solve Poisson's equations for pressure in hydrodynamic problems with incompressible media are the methods of converting Poisson's equations into a form containing only the pressure gradient and the method of converting the equations into a weak form containing no derivatives of the functions to be solved. Such methods allow obtaining a solution for two- and three-dimensional space but require a large amount of CPU power. As a rule, multivariate Poisson's equations can be solved using various approximation schemes. In this case, the solution is usually reduced to sweep methods or to setting problems for which a fictitious time is introduced [31].

All authors have noted that the solution requires a rather long iterative process and a large amount of CPU power. Therefore, the development of a solution to hydrodynamic problems using non-primitive variables, as well as justifying the possibilities of applying these solutions to direct and inverse hydrodynamic problems, will significantly expand the applied calculations.

## 3- Research Methodology

The full hydrodynamic equations written in "non-primitive variables" are transformed into an algebraic analogue using finite difference schemes. The evolutionary form of the vortex vector transport equation enables the use of the establishment method [2] with an output to a stationary solution. At each iterative time step of the computational evolution, the inverse hydrodynamics problem was solved. The velocity vector was calculated from the calculated vortex value and the given divergence of the velocity vector. This velocity vector was used in the next time step of the iterative computation process, and the new vortex position of the velocity vector field was calculated from it.

A fully conservative scheme [24, 32] is used to calculate vortex transport. The code was written in the GAMS language using SOLVER-CONOPT. The results were automatically output to an Excel spreadsheet file. The script within the spreadsheets generated velocity vector fields, showing the calculated velocity vector fields with arrows.

The calculations were performed on test cases where the nature of the solution was either obvious or where results obtained by independent researchers were available for comparison.

### *3-1- Transition to "Non-Primitive Variables"*

In order to ensure that no details are overlooked, the following section examines the transition to non-primitive variables in great detail. At each step of the transformation, the validity of the transformation and all assumptions made in the process will be monitored. This will avoid inexplicable computational "surprises" in the test and then practically necessary computational tools.

Let us consider the following basic equations of continuum fluid dynamics [3, 18-20]:

$$\begin{aligned}
&\frac{dV_x}{dt} = \frac{\partial(V_x)}{\partial t} + V_x\frac{\partial(V_x)}{\partial x} + V_y\frac{\partial(V_x)}{\partial y} + V_z\frac{\partial(V_x)}{\partial z} = -\frac{1}{\varrho}\cdot\frac{\partial P}{\partial x} + \nu\cdot\Delta V_x + f_x\\
&\frac{dV_y}{dt} = \frac{\partial(V_y)}{\partial t} + V_x\frac{\partial(V_y)}{\partial x} + V_y\frac{\partial(V_y)}{\partial y} + V_z\frac{\partial(V_y)}{\partial z} = -\frac{1}{\varrho}\cdot\frac{\partial P}{\partial y} + \nu\cdot\Delta V_y + f_y\\
&\frac{dV_z}{dt} = \frac{\partial(V_z)}{\partial t} + V_x\frac{\partial(V_z)}{\partial x} + V_y\frac{\partial(V_z)}{\partial y} + V_z\frac{\partial(V_z)}{\partial z} = -\frac{1}{\varrho}\cdot\frac{\partial P}{\partial z} + \nu\cdot\Delta V_z + f_z\\
&\frac{\partial\varrho}{\partial t} + V_x\cdot\frac{\partial\varrho}{\partial x} + V_y\cdot\frac{\partial\varrho}{\partial y} + V_z\cdot\frac{\partial\varrho}{\partial z} + \varrho\cdot\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)
\end{aligned} \tag{1}$$

or in vector form:

$$\begin{aligned}
&\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \vec{f} - \frac{1}{\varrho}\cdot\mathrm{grad}\,P + \nu\cdot\Delta\vec{V}\\
&\frac{\partial\varrho}{\partial t} + \mathrm{div}\varrho\vec{V} = \frac{\partial\varrho}{\partial t} + \overrightarrow{V}\cdot\mathrm{grad}\,\varrho + \varrho\cdot\mathrm{div}\vec{V} = 0
\end{aligned} \tag{2}$$

where: $\vec{V}$ is vector of motion of a continuous medium with components $V_x, V_y, V_z$; $d, \partial$ are full and partial derivatives; $x$, $y$, $z$ are spatial coordinates; $P$ is pressure; $\varrho$ is density of continuous medium; $\vec{f}$ is mass force with components $f_x, f_y, f_z$; $\nu$ is kinematic viscosity; $\mathrm{grad}(*), div(*), \nabla$ and $\Delta$are differential operators: gradient, divergence, symbolic vector-operator Hamiltonian (a synonym of Nabla-Operator) and Laplacian respectively.

Let us use equality linking the full and partial derivatives by the following relation:

$$\frac{d(\_)}{dt} = \frac{\partial(\_)}{\partial t} + V_x\frac{\partial(\_)}{\partial x} + V_y\frac{\partial(\_)}{\partial y} + V_z\frac{\partial(\_)}{\partial z} = \frac{\partial(\_)}{\partial t} + \vec{V}\cdot grad(\_) = \frac{\partial(\_)}{\partial t} + \vec{V}\cdot\nabla(\_) \tag{3}$$

The use of Hamiltonian and Laplacian allows the reduction and simplifying of the notation of vector equations.

For the certainty and transparency of the following entries and transformations, let us write down the formulations:

$$\begin{aligned}
&\nabla = \frac{\partial}{\partial x}\cdot\vec{i} + \frac{\partial}{\partial y}\cdot\vec{j} + \frac{\partial}{\partial z}\cdot\vec{k}, \quad \Delta = \nabla\cdot\nabla = \nabla^2 = \frac{\partial^2}{(\partial x)^2} + \frac{\partial^2}{(\partial y)^2} + \frac{\partial^2}{(\partial z)^2};\\
&\nabla b = grad\ b = \frac{\partial b}{\partial x}\vec{i} + \frac{\partial b}{\partial y}\vec{i} + \frac{\partial b}{\partial z}\vec{k}\\
&\nabla\cdot\vec{a} = div\ \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z};\\
&\nabla\times\vec{a} = rot\ \vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{k}
\end{aligned} \tag{4}$$

where: $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along OX, OY, OZ axes respectively; $b$ is some scalar quantity; $\vec{a}$, $(a_x, a_y, a_z)$ is some vector and its three components; $rot(\vec{a})$ differential operator – "vortex of $\vec{a}$ vector", with components $\omega_x, \omega_y, \omega_z$.

Let us assume that the value of density of the continuous medium is much larger than the possible changes of this density. Let us call such a medium weakly compressible.

That is $\partial\varrho < \varrho$.

This statement is very valid for water and even air in normal atmospheric processes. This statement allows the derivative of the value to be considered negligibly small.

Indeed,

$$\partial\left(\frac{1}{\varrho}\right) = -\left(\frac{1}{\varrho}\right)^2\partial\varrho \sim 0 \tag{5}$$

Therefore, it can be neglected and not even written down all terms containing the derivative of the medium density in the following transformations. The actions below are performed:

- Differentiate the third equation of system (1) by Y and the second equation (1) by Z and subtract the second result from the first one;

- Differentiate the first equation of system (1) by Z and the third equation (1) by X and subtract the second result from the first one;
- Differentiate the second equation of system (1) by X and the first equation (1) by Y and subtract the second result from the first one.

$$\frac{d\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)}{d t}=+\nu \cdot\Delta\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)$$
$$\frac{d\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}}{d t}=+\nu \cdot\Delta\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right) \tag{6}$$
$$\frac{d\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)}{d t}=+\nu \cdot\Delta\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)$$

Alternatively, given Equation 4, Equation 7 can be written as follows:

$$\frac{d\omega_x}{d t}=\nu \cdot\Delta\,\omega_x=\frac{\partial\omega_x}{\partial t}+\mathrm{V_x}\frac{\partial\omega_x}{\partial x}+\mathrm{V_y}\frac{\partial\omega_x}{\partial y}+\mathrm{V_z}\frac{\partial\omega_x}{\partial z}=\nu \cdot\Delta\,\omega_x$$
$$\frac{d\omega_y}{d t}=\nu \cdot\Delta\,\omega_y=\frac{\partial\omega_y}{\partial t}+\mathrm{V_x}\frac{\partial\omega_y}{\partial x}+\mathrm{V_y}\frac{\partial\omega_y}{\partial y}+\mathrm{V_z}\frac{\partial\omega_y}{\partial z}=\nu \cdot\Delta\,\omega_y \tag{7}$$
$$\frac{d\omega_z}{d t}=\nu \cdot\Delta\,\omega_z=\frac{\partial\omega_z}{\partial t}+\mathrm{V_x}\frac{\partial\omega_z}{\partial x}+\mathrm{V_y}\frac{\partial\omega_z}{\partial y}+\mathrm{V_z}\frac{\partial\omega_z}{\partial z}=\nu \cdot\Delta\,\omega_z$$

In vector form, expression (7) is even more simplified:

$$\frac{\partial\, rot(\vec{V})}{\partial t}+\left(\vec{V}\cdot\nabla\right) rot(\vec{V})=\nu \cdot\Delta\, rot\,(\vec{V}) \tag{8}$$

The velocity field should be calculated using the well-known relation (9) from field theory:

$$\Delta\,\vec{V}=-rot\left[rot\,(\vec{V})\right]+grad\;div\,\vec{V} \tag{9}$$

The system of Equations 8 and 9 forms a special kind of a record of the equations of motion of a compressible continuous medium with the so-called "non-primitive variables": $div\,\vec{V}$ and $rot(\vec{V})$. Vector Equation 9 is called "inverse hydrodynamic problem". Here, the vector field of velocity is determined by the known vector field of the vortex $rot(\vec{V})$ and the scalar divergence field $div\,\vec{V}$.

Let us note the two most important features and peculiarities of the system of Equations 8 and 9.

1. There is no variable "pressure $P$ in a continuous medium".
2. Equation 9 is not a consequence of the extinct and unnecessary Equation 2. Equation 9 is the simplest relationship of any vector field with its "vortex" and "divergence". This means that the system of Equations 8 and 9 is applicable to the calculation of displacements of weakly compressible media and does not contain any strict restrictions on incompressibility.

### *3-2- Statement of the Problem on the Flow Motion of a Coherent Medium*

The solution to a test problem involving the flow of a coherent medium with a conservative admixture in a region with irregular obstacles is evaluated for its economy and adequacy. If possible, the results will be compared with those obtained by independent authors. This problem is of great practical importance as it helps to analyze the distribution of pollution in urban development or the distribution of harmful discharges in rivers and canals. The problem will be solved in a two-dimensional flat horizontal space, assuming that there are no fundamental restrictions on calculations in three-dimensional space, given the presence of powerful computing equipment. Note that in three-dimensional space, it is possible to calculate unsteady processes with a small change in the density of the moving medium.

It follows from literary sources that most authors using "unprimitive" variables $rot(\vec{V})$ and $div\,\vec{V}$, when solving two-dimensional problems, refuse to use Equation 9 [27, 33, 34]. The rejection is explained by the fact that equation (9) is vectorial, and even in the case of a two-dimensional problem, two components remain. Each component is a Poisson equation, the solution of which presents certain difficulties. Therefore, a way has been developed in which instead of two Poisson equations, only one can be solved by defining a special new variable, the potential $\psi_0$ [9, 22].

This approach requires the introduction of restrictions on the processes of motion: either the medium is assumed to be absolutely incompressible, or (which has very rarely been calculated) the motion is assumed to be steady-state. The latter assumption generates a serious problem of the subsequent separation of matter flows into velocity and density. In any case, how it is done and whether it is done at all could not be found in literary sources.

In two-dimensional space, Equation 7 is simplified as follows: for the horizontal plane, only the vertical component $\omega_z$ of the "vortex" vector remains valid. The two components of the vector vortex transfer equation will disappear, and the remaining equation will take the form Equation 10:

$$\frac{\partial \omega_z}{\partial t} + V_x \frac{\partial \omega_z}{\partial x} + V_y \frac{\partial \omega_z}{\partial y} = \nu \cdot \Delta\, \omega_z \quad (10)$$

Furthermore, the equation connecting the potential field $\psi$ with the vertical component of the "vortex" vector (11) and expressions connecting the potential with the components of the velocity vector (12) are adopted in the solution.

$$\Delta\, \psi = -\omega_z \quad (11)$$

$$V_x = \frac{\partial \psi}{\partial y}\ ;\ \ V_y = -\frac{\partial \psi}{\partial x} \quad (12)$$

Equation 10 is a consequence of the definition of a new variable - potential ψ. The relationship between the potential and the velocity vector is defined by Equation 11. The Poisson Equation 11 can be obtained by differentiating the expression for $V_x$ by ∂y in Equation 12 and the expression for $V_y$ by ∂x in Equation 12, and then finding the difference between the results.

The property of incompressibility of the medium is fulfilled automatically and is not difficult to check. It is sufficient to differentiate the expression for $V_x$ by $\partial x$ in (12) and the expression for $V_y$ by $\partial y$ in Equation 12, and add up the results. It should be noted that the incompressibility property is not fundamental for the application of law (15) in the calculations of steady-state processes [35]. The main limitation of the current function is that it can only be defined in two-dimensional space.

However, if the system of Equations 8 and 9 is used to solve the problem, all restrictions on the use of non-primitive variables for weakly compressible fluids in three-dimensional space will be eliminated. This will enable calculations to be performed for weakly compressible media even under unsteady conditions.

### *3-3- Determination of the Initial and Boundary Conditions*

***Initial Conditions*:** The initial condition is required only for the vortex field in Equation 8 [18].

$$\omega_z \big|_{t=0} = \Omega_o \quad (13)$$

***Boundary Conditions*:** The boundary conditions for the velocity vector field and the potential on the solid impermeable boundaries of the solution region are written as follows:

$$V_\tau \big|_G = \frac{\partial \psi}{\partial n} \big|_G = 0 \ \ and\ \ V_n \big|_G = -\frac{\partial \psi}{\partial \tau} \big|_G = 0 \quad (14)$$

The boundary conditions for the velocity vector field at the section of the continuous medium inflow into the calculation area are set as the known velocity vector $Vo_\tau, Vo_n$ epure and the potential value ψ and the vortex value $\omega_z$ are calculated from it.

$$V_\tau \big|_{G_{in}} = \frac{\partial \psi}{\partial n} \big|_{G_{in}} = Vo_\tau \ \ and\ \ V_n \big|_{G_{in}} = -\frac{\partial \psi}{\partial \tau} \big|_{G_{in}} = Vo_n \quad (15)$$

$$\omega_z \big|_{G_{in}} = \frac{\partial V_\tau}{\partial n} - \frac{\partial V_n}{\partial \tau} \quad (16)$$

where: $n$ is a unit vector perpendicular to the boundary; $\tau$ is a unit vector tangential to the boundary.

The boundary conditions for the velocity vector field on the outflow section of the continuous medium in the calculation domain are set in a form that allows smooth conjugation of the obtained solutions to the situation on the boundary.

$$\frac{\partial \psi}{\partial n} \big|_{G_{out}} = 0\ and \quad \frac{\partial \omega_z}{\partial n} \big|_{G_{out}} = 0 \quad (17)$$

### *3-4- Finite-Difference Analog*

Equations 10 to 12 can be written in finite-difference form by the following algebraic analog:

$$\begin{gathered}\frac{\omega_{i,j}^{t+1}-\omega_{i,j}^{t}}{\Delta t}\\ +\max(0,V_{i,j}^{t})\frac{\omega_{i,j}^{t}-\omega_{i-1,j}^{t}}{\Delta x}+\omega_{i,j}^{t}\frac{\max(0,V_{i+1,j}^{t})-\max(0,V_{i,j}^{t})}{\Delta x}\\ +\min(0,V_{i,j}^{t})\frac{\omega_{i+1,j}^{t}-\omega_{i,j}^{t}}{\Delta x}+\omega_{i,j}^{t}\frac{\min(0,V_{i,j}^{t})-\min(0,V_{i-1,j}^{t})}{\Delta x}\\ +\max(0,U_{i,j}^{t})\frac{\omega_{i,j}^{t}-\omega_{i,j-1}^{t}}{\Delta y}+\omega_{i,j}^{t}\frac{\max(0,U_{i,j+1}^{t})-\max(0,U_{i,j}^{t})}{\Delta y}\\ +\min(0,U_{i,j}^{t})\frac{\omega_{i,j+1}^{t}-\omega_{i,j}^{t}}{\Delta y}+\omega_{i,j}^{t}\frac{\min(0,U_{i,j}^{t})-\min(0,U_{i,j-1}^{t})}{\Delta y}\\ +\nu\cdot\left(\frac{\omega_{i+1,j}^{t}-2\cdot\omega_{i,j}^{t}-\omega_{i-1,j}^{t}}{\Delta x^{2}}+\frac{\omega_{i,j+1}^{t}-2\cdot\omega_{i,j}^{t}-\omega_{i,j-1}^{t}}{\Delta y^{2}}\right);\\ \frac{\psi_{i+1,j}^{t}-2\,\psi_{i,j}^{t}-\psi_{i-1,j}^{t}}{\Delta x^{2}}+\frac{\psi_{i+1,j}^{t}-2\,\psi_{i,j}^{t}-\psi_{i-1,j}^{t}}{\Delta y^{2}}=-\omega_{i,j}^{t};\\ V_{i,j}^{t}=\frac{\psi_{i+1,j}^{t}-\psi_{i-1,j}^{t}}{2\cdot\Delta y};\qquad U_{i,j}^{t}=-\frac{\psi_{i,j+1}^{t}-\psi_{i,j-1}^{t}}{2\cdot\Delta x}\end{gathered}\tag{18}$$

When constructing a finite-difference analog, a finite-difference scheme was used to calculate the terms responsible for calculating the transport in the substance or momentum space. The scheme was developed at the Tashkent Institute of Irrigation and Agricultural Mechanization Engineers National Research University (TIIAME NRU) [24, 32]. In finite-difference and algebraic equations, the symbol Δ was traditionally used to denote a segment on a coordinate axis (not to be confused with the Laplacian differential operator). The scheme has conservativity, stability, transportability, and adequacy for any configuration of the velocity field, and it also has the property of invariance. The presence of all these properties simultaneously distinguishes it from other schemes currently used.

## 4- Solution of the Test Problems

### *4-1- Problem 1*

Test Problem 1 demonstrates the identity of the obtained solutions of the full system of equations of motion in the ‘nonprimitive’ variables (8) and (9) and the approach involving a new variable ‘current function’. The test problem is simple to ensure the identity of initial and boundary conditions in both systems of equations and to show exactly the identity of solutions. In a flat rectangular region, set a vortex other than zero at one of the points near the center of the region. The space step is 1 m, and the time step is 0.5 s. The kinematic viscosity coefficient is set to 0.01 $m^2$/s. The vortex at the initial moment of time is 0.11 m/s. Figure 1 shows the velocity distribution around the vortex at the 5$^{th}$ second of the process. Note that if the kinematic viscosity is set equal to zero, the velocity distribution diagram stops changing over time, and the vortex remains stable. This is consistent with Helmholtz’s fourth theorem on the conservation of vorticity in an ideal medium [19, 20].

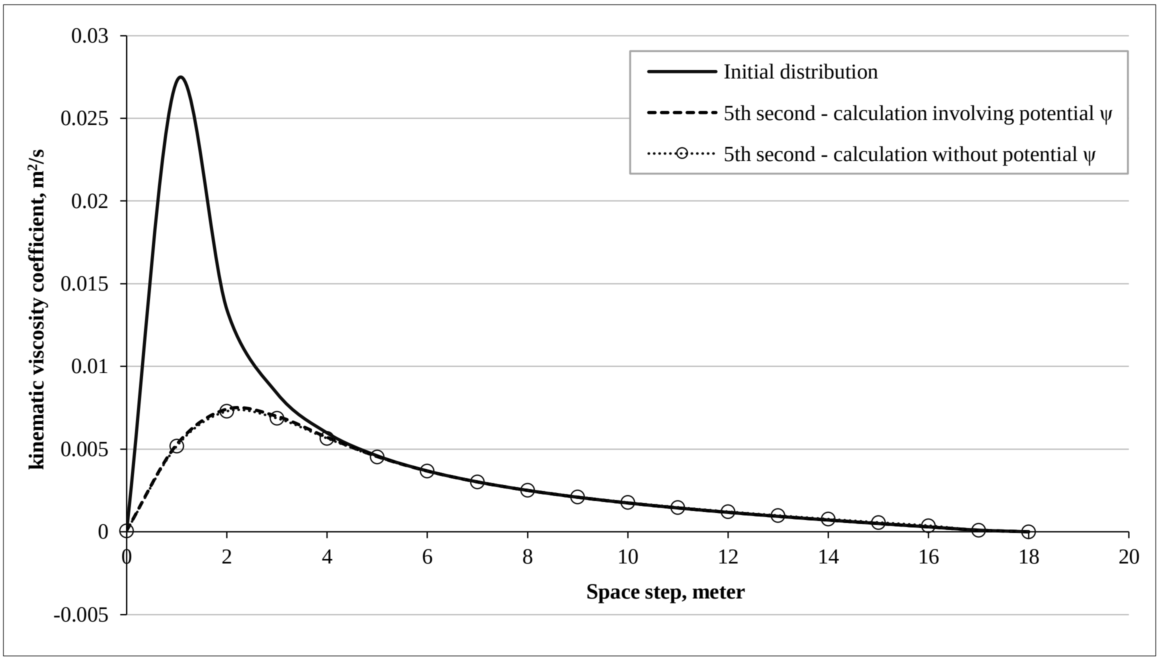


**Figure 1. Velocity profile in a single vortex and vortex dissipation process**

The solution of problem 1, set to work out the calculation algorithm, showed the excellent computational quality of the scheme developed at TIIAME NRU [24, 32] for calculating the convection momentum transfer and conservative admixtures.

Figure 2 shows the velocity vector field at 5 s of vortex dissipation. Note an important fact: the result of the solution of systems (8) and (9) in two-dimensional space (without the variable ψ) coincides with the result of the solution of systems (10), (11), and (12) (with the variable ψ). The coincidence of solutions of two different systems of equations is shown in Figure 1. Calculation was carried out under the condition that the medium is incompressible, because otherwise it would not be possible to use the system of Equations 10 to 12. Thus, the restrictions on the calculation of the motion of a solid medium in two-dimensional space set by systems (10), (11), and (12) are not fundamental. Always in case of necessity, it is possible to pass to the solution of system (8), (9), although such a transition will require almost twice as much for calculations.

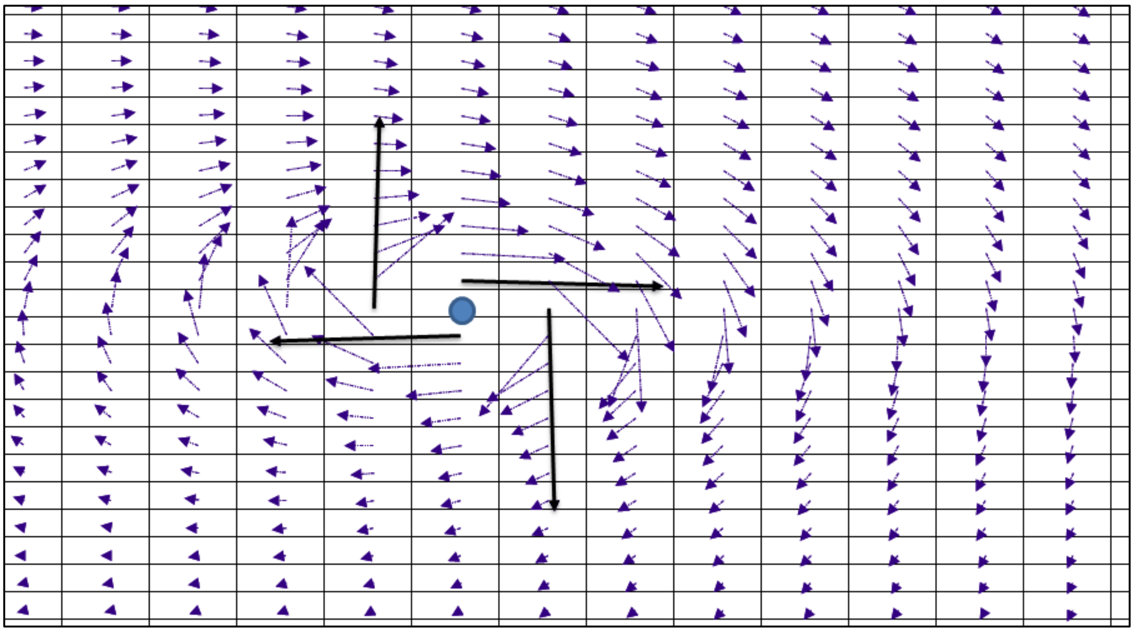

**Figure 2. Field of the velocity vector of a single vortex at the 5th second**

### *4-2- Problem 2*

Test Problem 2 demonstrates the capabilities of the full system of equations of motion in 'nonprimitive' variables (8) and (9) to describe the interactions of multiple vortex formations within the computational domain. Two chains of differently directed vortices are located in a rectangular area. Figure 3 shows two chains of five vortices each. If in the upper chain all vortices rotate in one direction, in the lower chain they rotate in the opposite direction.

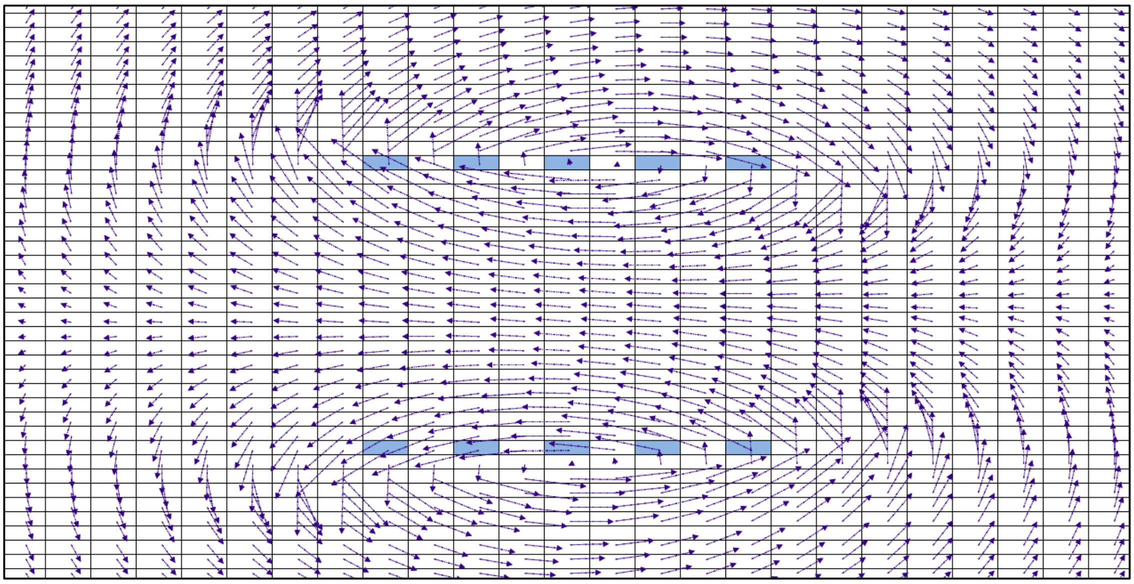

**Figure 3. Scheme of motion of a continuous medium formed by two chains of differently directed vortices**

Let us determine the field of velocity that will form these two chains of vortices if the medium is incompressible. The direction of the vortices in the chains is determined by the right-hand borax rule. For example, the upper chain of vortices (Figure 3) rotates clockwise (the right borax rotates clockwise), and consequently, the vortices of the upper chain are directed beyond the plane of Figure 3. In the lower chain, the vortices are directed in the opposite direction. Figure 4 shows the velocity diagrams in the cross-section crossing both vortex chains. The matter flow between the vortex chains is completely compensated for by the matter flow around the vortex chains on the outer side. This is quite consistent with the previously accepted assumption of incompressibility of the medium.

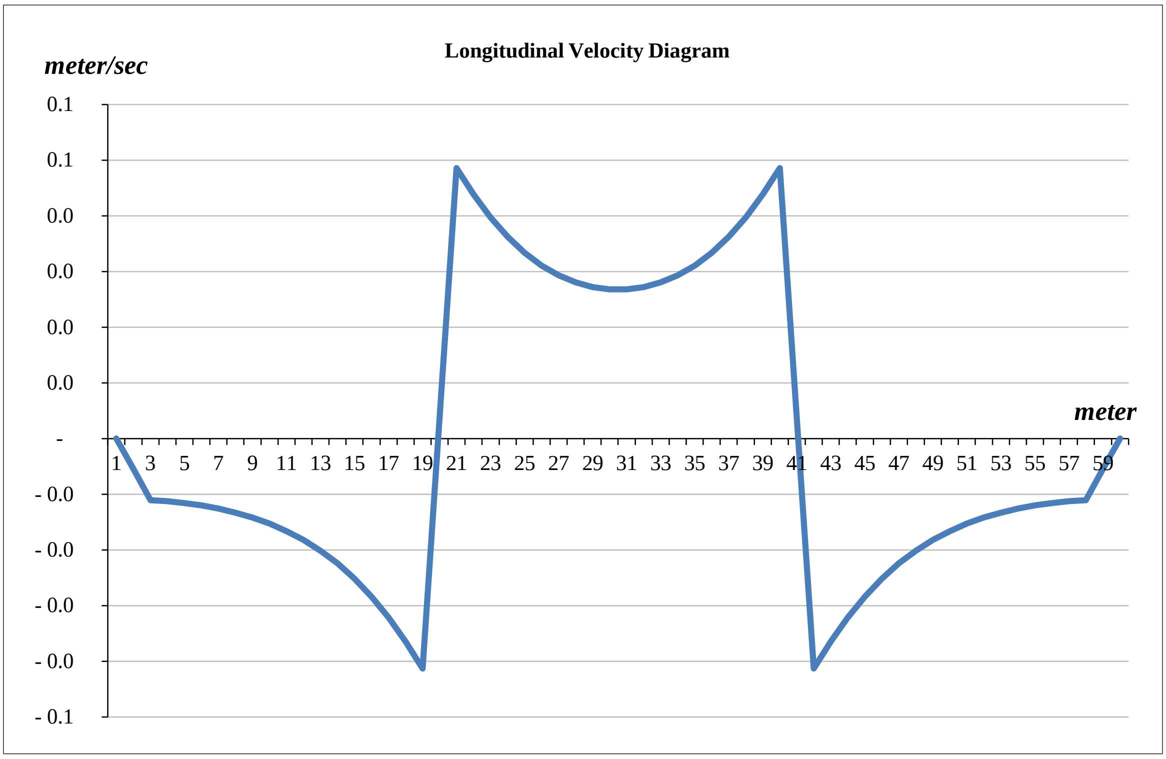


**Figure 4. Velocity diagram in the cross-section perpendicular to the location of the chains of multidirectional vortices**

Most practically important fluid dynamics problems are related not only to the determination of velocity fields but also to the transport of impurities in a moving gas or liquid flow. Once the algorithm for finding the velocity vector field based on the given vortex field and divergence field has been found and tested in test problems 1 and 2, it can proceed to the test problem in which the impurity transport from some source is calculated along with the velocity field. However, for this purpose, it is necessary to supplement the system of equations defining flow velocity with an equation defining admixture transport.

The scheme developed in TIIAME NRU for calculating the momentum transfer and vortices allows solving the problem of conservative admixture transport in a channel with a complex configuration of banks. In order to solve this problem, let us supplement the system of Equations 8 and 9 with the admixture conservation Equation 19.

$$\frac{\partial S}{\partial t} + \overrightarrow{V} \cdot \text{grad}\, S + S \cdot \text{div}\vec{V} = D_s \cdot \triangle S + I_o \tag{19}$$

where $S$ is the concentration of some conservative impurity and $I_o$ is the impurity entering the stream from a source (discharge from an industrial plant into a river, gas emissions from a road, or some other source).

The boundary conditions are set as follows:

At the impermeable boundary and the outflow zone $G_{out}$ and along the perpendicular vector $n$ at this boundary.

$$\frac{\partial S}{\partial n} \Big|_{G_{out}} = 0 \tag{20}$$

In the inflow zone $G_{in}$, the impurity concentration $S_t$ is given, which generally depends on time $t$.

$$S \Big|_{G_{in}} = S_t \tag{21}$$

The initial condition is to set the field of concentration of the admixture at the initial moment of time $S_0$.

$$S \Big|_{t=0} = S_o \tag{22}$$

Applying the TIIAME NRU scheme, let us write the finite-difference analogue by an algebraic equation [33, 34].

$$
\begin{aligned}
&\frac{S_{i,j}^{t+1}-S_{i,j}^{t}}{\Delta t}= \\
&+\max\left(0,V_{i,j}^{t}\right)\frac{S_{i,j}^{t}-S_{i-1,j}^{t}}{\Delta x}+S_{i,j}^{t}\frac{\max\left(0,V_{i+1,j}^{t}\right)-\max\left(0,V_{i,j}^{t}\right)}{\Delta x} \\
&+\min\left(0,V_{i,j}^{t}\right)\frac{S_{i+1,j}^{t}-S_{i,j}^{t}}{\Delta x}+S_{i,j}^{t}\frac{\min\left(0,V_{i,j}^{t}\right)-\min\left(0,V_{i-1,j}^{t}\right)}{\Delta x} \\
&+\max\left(0,U_{i,j}^{t}\right)\frac{S_{i,j}^{t}-S_{i,j-1}^{t}}{\Delta y}+S_{i,j}^{t}\frac{\max\left(0,U_{i,j+1}^{t}\right)-\max\left(0,U_{i,j}^{t}\right)}{\Delta y} \\
&+\min\left(0,U_{i,j}^{t}\right)\frac{S_{i,j+1}^{t}-S_{i,j}^{t}}{\Delta y}+S_{i,j}^{t}\frac{\min\left(0,U_{i,j}^{t}\right)-\min\left(0,U_{i,j-1}^{t}\right)}{\Delta y} \\
&+D_s\cdot\left(\frac{S_{i+1,j}^{t}-2\cdot S_{i,j}^{t}-S_{i-1,j}^{t}}{\Delta x^2}+\frac{S_{i,j+1}^{t}-2\cdot S_{i,j}^{t}-S_{i,j-1}^{t}}{\Delta y^2}\right);
\end{aligned}
\tag{23}
$$

Using the transport and conservation of momentum, matter and "vortex" scheme developed at TIIAME NRU [24, 32], it is possible to make practically meaningful calculations about the distribution of impurities and pollutants entering water flows and moving layers of the atmosphere, which have been considered by Marchuk [2], Lax and Richtmyer [23], Popov and Timofeeva [14], Marchuk [2], Batchelor & Moffat [21].

### *4-3- Problem 3*

Problem 3 demonstrates the system's ability to compute complex flow configurations using non-primitive variables in the full form of Equations 8 and 9.

Two streams of water (air) flow into a rectangular area. Next to one of them, some pollutant constantly enters. It is necessary to determine the velocity field and the field of spreading of the pollutant.

Figure 5 shows the velocity vector field, and Figure 6 shows the longitudinal velocity distribution. The left embedded plot in Figure 6 shows the longitudinal velocity distribution. The right embedded plot shows the law of conservation of the mass of the moving medium from transverse to transverse in the calculation area.

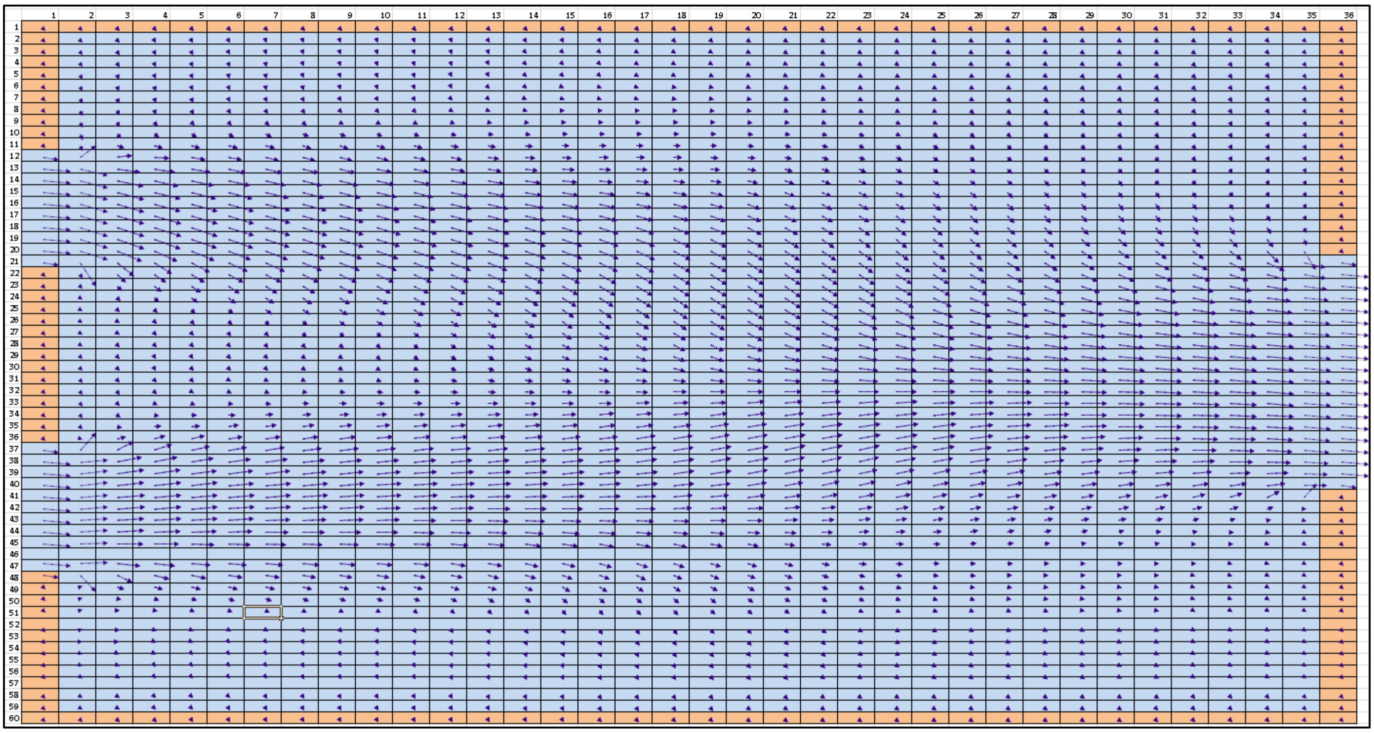

**Figure 5. Velocity vector field obtained by solving problem 3**

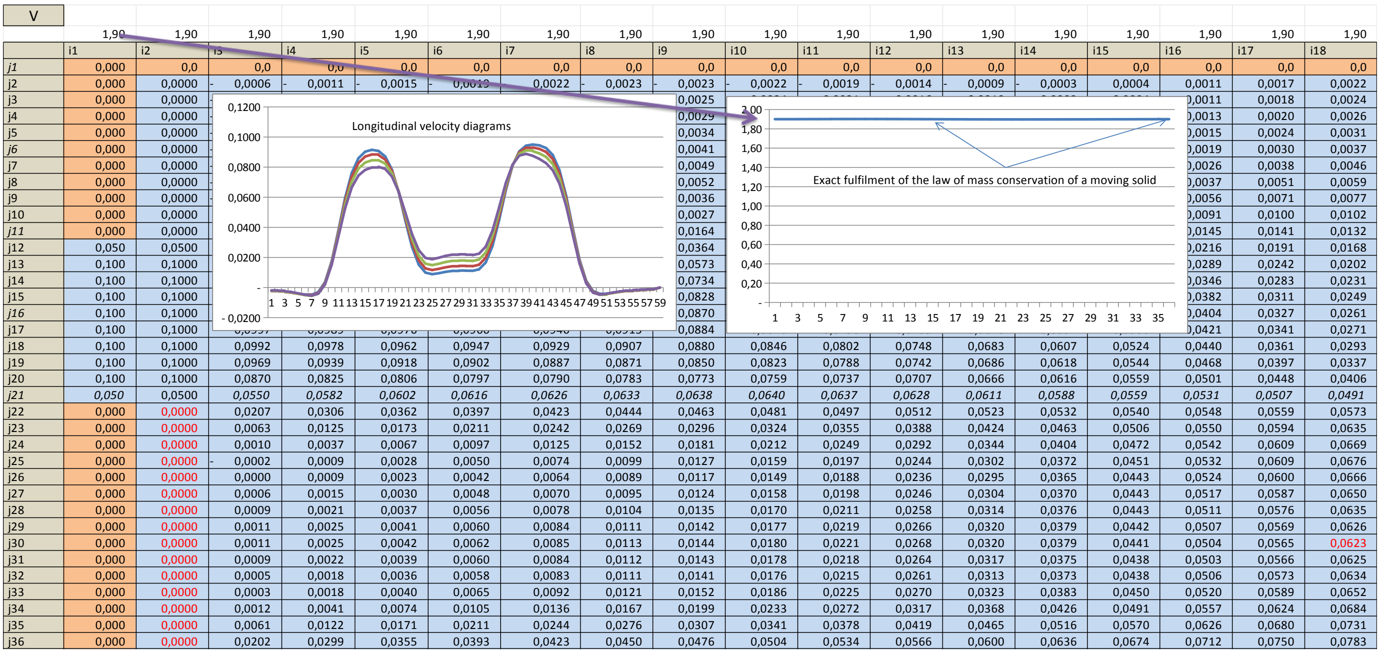


**Figure 6. Longitudinal velocity diagrams obtained in problem 3**

The demonstration of reaching the stationary state of the conservative admixture content at time step 450 (the 2250$^{th}$ second of the physical process) is shown in Figure 7. The amount of matter entering the calculation zone is exactly equal to the amount of matter leaving the calculation zone. The line becomes almost horizontal after the 400$^{th}$ calculation step.

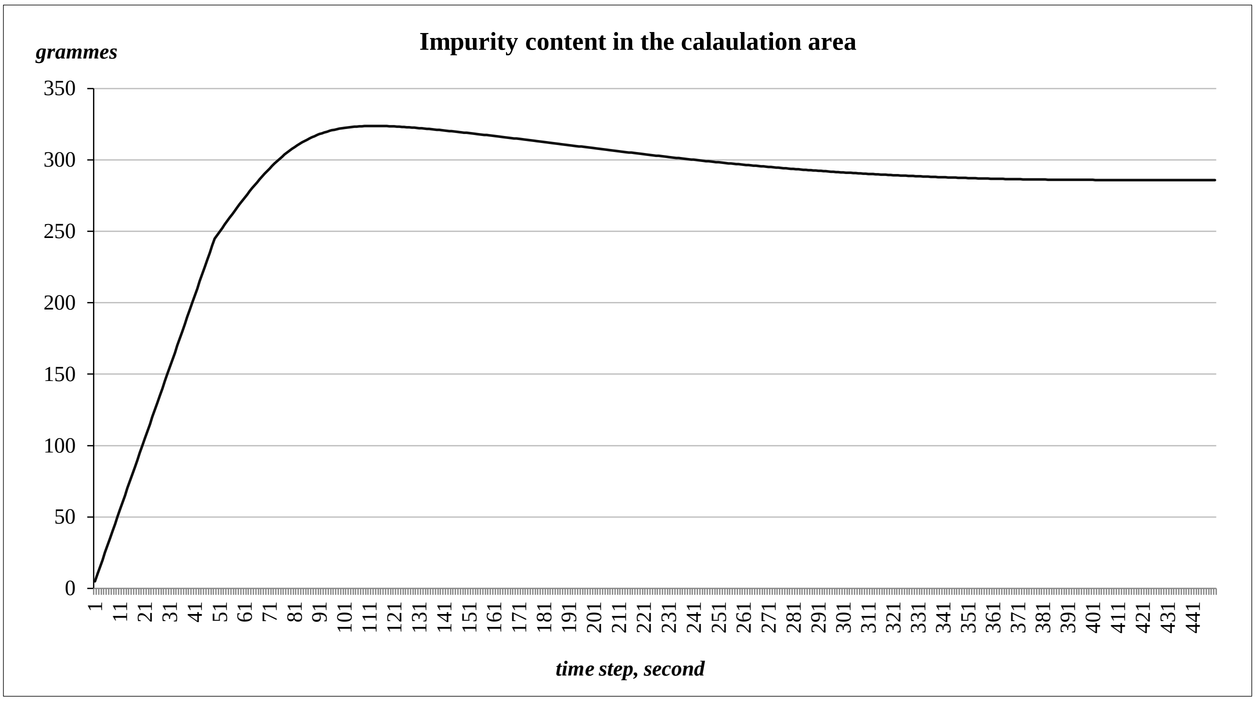


**Figure 7. Exit to the steady-state value of the conservative impurity content in the stream**

The check showed that the water balance (longitudinal and cross-sectional) for each cross-section of the calculation area is very accurate. This balance is observed in all time steps starting from the first one. An exact balance in the presence and movement of the pollutant is also performed.

### *4-4- Problem 4*

Problem 4 demonstrates the possibility of solving the system of equations in 'non-primitive' variables (8) and (9) in a region with complex boundary shapes. The problem of propagating impurities in a moving flow is solved using finite-difference schemes [24, 32], which guarantee the accuracy of the conservation laws of the impurity and velocity vector vortex.

The solution to the inverse problem of hydrodynamics, even in complex shapes, ensures the exact conservation of the mass of the moving medium carrying the impurity. It is important to note that the accuracy of conservation laws is not always guaranteed when applying known schemes [3, 10, 23] to convergent and divergent flows [22, 24, 32]. Two flows (water or air) enter a rectangular area. Impermeable obstacles (islands or buildings) exist on the path of the flow. There is a point where a substance dissolved in water or gas from some object enters the moving stream. All that needs to be done to organize such a calculation is to ensure that the potential isoline coincides with the boundary of the impermeable object.

Figure 8 shows the flow pattern in a calculation area containing two impermeable objects. The vortex motion organized around the right-hand impermeable object is very interesting

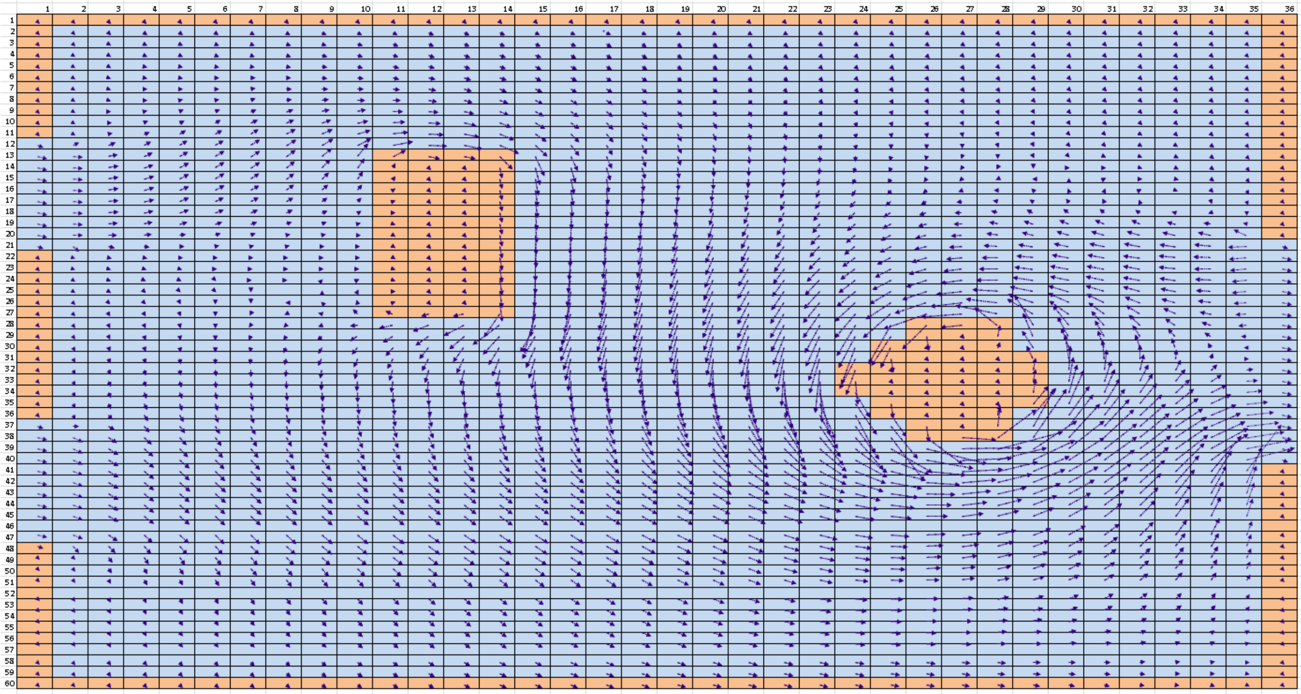

**Figure 8.** **Schematic of the motion of a continuous medium with two impermeable obstacles in the flow area**

The calculation results are shown in Figure 9, which shows the flow contamination field from a point source. The point source location is marked in bright yellow.

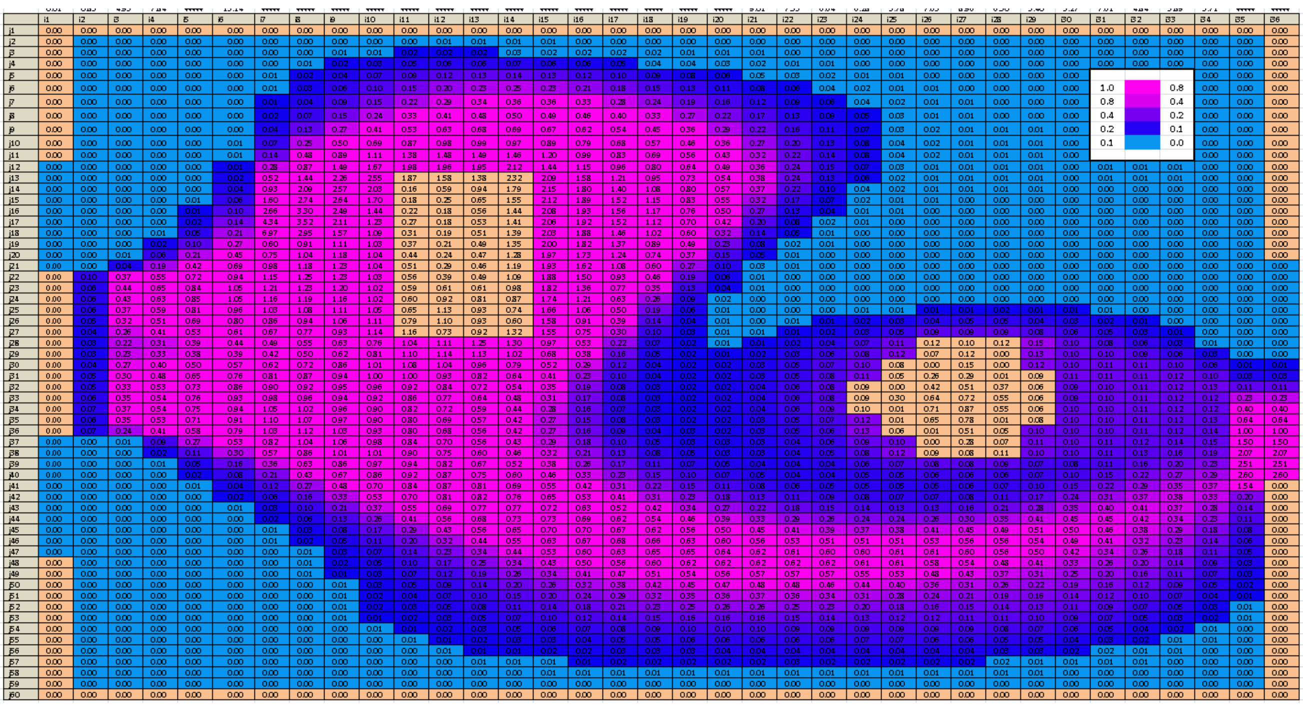

**Figure 9.** **Propagation field of impurities entering the calculation area from a point**

The following features of the calculation performed were noted:

- Exact implementation of the water balance. Through each cross section, the flow is equal to the sum of two flows entering the calculation area to the left.
- Exact implementation of the mass conservation law for impurities.

No wave processes that strongly limit the possibility of solving similar problems when using primitive variables – “velocity” and “pressure”

## 5- Discussion

There are few studies that deal with hydrodynamic equations written in non-primitive variables. Typically, research works focus on a single test problem [4, 17, 31]. Usually, only one or two vortex formations are considered [4, 6, 16]. All published works on “non-primitive” variables have a commonality. They are related by the definition of a new variable - the current function. The current function strictly limits the dimensionality of hydrodynamic problems to two dimensions. In addition, the current function is only applicable to the study of incompressible medium motions. The compressibility of the medium undermines the definition of the variable current function [18]. Researchers focused on studying the variable current function because it allowed them to solve one Poisson equation instead of three equations in the three-dimensional domain when solving the inverse problem of fluid dynamics. This was partly due to the limited computing power and programming language capabilities available at the time. However, the situation has changed with the availability of fast computers and programming languages capable of independently algorithmizing calculations.

Therefore, the investigation of the basic and general problem of hydrodynamics in ‘non-primitive’ variables is of research interest and practical engineering feasibility.

This paper provides a detailed construction of the equations of hydrodynamics in ‘non-primitive’ variables to describe the motion of a compressible medium. The purpose of this study is to demonstrate the assumptions required to obtain the well-known equations [19, 20]. The medium’s small relative compressibility and density, expressed by a number equal to or greater than unity (5), are the primary assumptions. It is highly unlikely to compactly write down and solve equations in “non-primitive” variables for sparse media with small densities. Condition (5) prohibits this.

The test problems were designed to have easily identifiable solutions or to allow comparison with results from other authors. The solutions to the test problems demonstrate the following: Solving the impurity propagation test problem in a complex region demonstrates the practical application of the obtained equations in hydrodynamics. This text discusses the correspondence between solutions from the full system of hydrodynamic equations in “nonprimitive” variables and practical observable phenomena, specifically the velocity fields of a single vortex or a chain of vortices. It also explores the identity of solutions of the full system of equations in “non-primitive” variables and the system of equations using the potential field, as demonstrated by the test problem of a single vortex.

The text discusses the potential use of the full system of equations in “non-primitive” variables to calculate the motion of interacting flows. One test problem demonstrates the coupling of two previously unconnected flows into one common flow. Additionally, the text highlights the possibility of obtaining solutions to an important engineering problem of impurity propagation in a region with a complex shape based on hydrodynamic equations in “non-primitive” variables.

As there are limited studies on dealing with non-primitive variables, one of the most widely studied problems in fluid dynamics was considered to compare the results obtained with those of other researchers. In particular, the problem of vortex formation occurring in a rectangular pocket at the boundary of a moving flow was considered when the flow moves along the upper slice of this pocket. The investigation results are presented in Yarmitskiy [25], while the calculations using non-primitive variables are described in Ivanov [6].

Figure 10 shows a comparison between the results obtained by solving the problem using the system of Equations 8 and 9 and those published in Ivanov [6] and Yarmitskiy [25]. A comparison was made for the horizontal and vertical velocities of vortex motion inside the ‘pocket’ on vertical and horizontal sections drawn through the center of the ‘pocket’. Based on calculations using Equations 8 and 9, the vortex center formed in the ‘pocket’ is most intense in the upper part and shifts toward the main flow. This is consistent with the findings of Ivanov [6]. The solution results are comparable in both qualitative and numerical characteristics to published data [6, 25].

Throughout the calculations, the laws of conservation of momentum and mass of the moving medium were verified. The laws of conservation of momentum and matter were found to be accurately fulfilled at every step of the iterative process and at any cross-section of the calculation domain in all test problems. This demonstrates the accuracy of the algorithms used to solve the problems and the chosen calculation methods [24, 32].

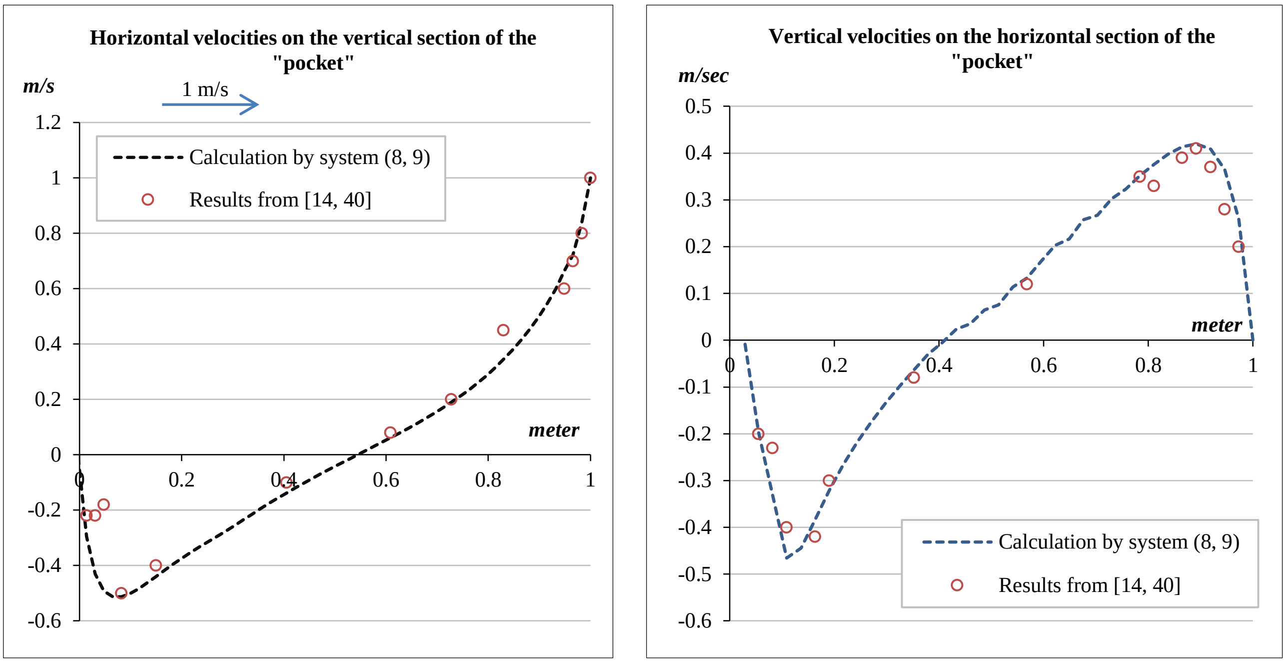


**Figure 10.** **Comparison of the results of the solution of the system of equations (8, 9) about the vertical and horizontal components of the velocity vector at the sections inside the rectangular pocket with the data published in the source [6, 25]**

## 6- Conclusion

Despite their promise, "non-primitive" variables are poorly understood in theoretical fluid dynamics and are almost never used in practical computational fluid dynamics. Nevertheless, equations with "non-primitive" variables continue to be studied. New publications on this topic are appearing. Compared to studies of equations in "primitive" variables, such publications are relatively few, but they do exist [17, 31], which emphasizes the existence of interest in this area of computational fluid dynamics. In this paper, the computational fluid dynamics equations written in "non-primitive" variables have been studied, and some hydrodynamic problems have been constructed and solved. The application of optimizing programming languages allowed us to partly solve the problems related to the algorithmization of the computational process. The finite-difference transfer scheme is described and tested in this paper; its application provided a stable solution to some test problems. This scheme has full conservativity for arbitrary distributions of changing velocity fields, which is the main factor in obtaining stable solutions.

The useful theoretical transformations of equations and the form of the main finite-difference schemes of conservation and transfer of momentum of motion and matter given in this paper allowed us to obtain the solution to the simplest and most complicated problems of hydrodynamics. This represents a significant contribution to modern research in the field of fluid dynamics. The problems considered are encountered in computational fluid dynamics, aerodynamics, and environmental engineering. The study of hydrodynamic equations in "non-primitive" variables and the search for efficient algorithms for their solution are of great scientific and practical value. At present, this area of research is underdeveloped, and therefore, there is a great prospect of obtaining interesting and useful results.

### *6-1- Strengths of the Study*

State-of-the-art computational SOLVERs and GAMS-level programming languages were used to achieve a fast solution of the three components of the three-dimensional Poisson equation and obtain vortex motions in both compressible and incompressible media. The research sector is greatly expanded by the ability to solve spatial problems related to the motion of a compressible medium. The hydrodynamic equations, presented in "non-primitive" variables and using the methods, programming languages, and schemes proposed in this paper, enable the resolution of more complex problems than previously achieved.

### *6-2- Limitations of the Study*

Although equations in "non-primitive" variables have been proven to be effective in solving practical problems related to impurity distribution while maintaining high accuracy and substance balance, some conceptual issues still need to be addressed.

One such issue is the conjugation problem between the pure vortex and the pure potential displacements of the medium, which has been identified in field theory studies and classical hydrodynamics. One such issue is the conjugation problem between the pure vortex and the pure potential displacements of the medium, which has been identified in field theory studies and classical hydrodynamics. This problem remains unsolved and requires further research.

## 7- Declarations

### 7-1- Author Contributions

Conceptualization, A.Sal. and A.Sav.; methodology, A.Sav. and M.R.; software, A.Sav. and O.A.; validation, T.K., A.Sav., and A.G.; formal analysis, M.R. and M.A.; investigation, M.A. and O.A.; data curation, K.S.; writing—original draft preparation, A.Sav., M.R., and K.S.; writing—review and editing, M.R. and A.Sal.; visualization, R.R.; Supervision, A.Sal.; project administration, A.Sal.; funding acquisition, A.Sal. All authors have read and agreed to the published version of the manuscript.

### 7-2- Data Availability Statement

The data presented in this study are available in the present article.

### 7-3- Funding

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### 7-4- Institutional Review Board Statement

Not applicable.

### 7-5- Informed Consent Statement

Not applicable.

### 7-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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