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# Topology of Black Holes' Horizons

Arturo Tozzi [a*], James F Peters [b]

[a] Center for Nonlinear Science, University of North Texas, 1155 Union Circle, #311427, Denton, TX 76203-5017, USA

[b] Department of Electrical and Computer Engineering, University of Manitoba, 75A Chancellor's Circle, Winnipeg, MB R3T 5V6, Canada

**Abstract**

The Möbius strip spacetime topology and the entangled antipodal points on black hole surfaces, recently described by 't Hooft, display an unnoticed relationship with the Borsuk-Ulam theorem from algebraic topology. Considering this observation and other recent claims which suggest that quantum entanglement takes place on the antipodal points of a $S^3$ hypersphere, a novel topological framework can be developed: a feature encompassed in an $S^2$ unentangled state gives rise, when projected one dimension higher, to two entangled particles. This allows us to achieve a mathematical description of the holographic principle occurring in $S^2$. Furthermore, our observations let us to hypothesize that a) quantum entanglement might occur in a four-dimensional spacetime, while disentanglement might be achieved on a motionless, three-dimensional manifold; b) a negative mass might exist on the surface of a black hole.

**Keywords:**

Borsuk-Ulam Theorem;
Antipodal Points;
Quantum Entanglement;
Holographic Principle;
t'Hooft;
Möbius Strip.

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## 1- Introduction

The spacetime topology of a black hole has been recently described in four dimensions [1, 2]. The impenetrable, continuous curtain surrounding the black hole, termed firewall, displays antipodal quantum states with matching description. This also means that particles emerging at opposite sides of the 4-dimensional hypersphere are strongly entangled. In turn, recent claims suggest that quantum entanglement can be assessed in terms of opposite features on a 4D hypersphere. Indeed, Peters and Tozzi [3, 4] showed that a separable state can be achieved for each of the entangled particles lying in $S^2$, just by embedding them in a higher dimensional $S^3$ space. The Authors view quantum entanglement as the simultaneous activation of signals in a 3D space mapped into a $S^3$ hypersphere. Because the particles are entangled at the $S^2$ level and un-entangled at the $S^3$ hypersphere level, a composite system is achieved, in which each local constituent is equipped with a pure state.

It is noteworthy that both the issues, i.e., the black hole's antipodal points and the entanglement on a hypersphere, are assessable through the framework described by the Borsuk-Ulam theorem (BUT), which states that e*very continuous map* $f: S^n \rightarrow R^n$ *must identify a pair of antipodal points* – diametrically opposite points on an n-sphere [5, 6]. Points are *antipodal*, provided they are diametrically opposite [7-9]. Examples of antipodal points are the endpoints of a line segment, or opposite points along the circumference of a circle, or poles of a sphere, or the opposite quantum states with matching description embedded in the 't Hooft's four-dimensional black hole surface [10-12]. In other words, the BUT states that two features with matching description are mapped to a single feature one dimension lower, provided the function under assessment is continuous. In the case of 't Hooft's account of black holes, the continuity is preserved, because the firewalls of their surfaces are continuous. In the sequel, we will show how the BUT is correlated with the holographic principle and will draw unexpected consequences.

* **CONTACT**: Tozziarturo@libero.it

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## 2- Holographic Principle and Topological Mappings

The possibility dictated by the BUT to proceed from higher to lower dimensions and vice versa leads us to the realm of the holographic principle (HP). It states that the description of a volume of space can be thought of as encoded on a lower-dimensional boundary to the region [13, 14]. The theory suggests that the entire universe can be seen as two-dimensional information on the cosmological horizon. In HP, information (albeit quantum states evolving in spacetime) can be represented as a hologram, explainable via the theory of topological deformation retracts [15]. A deformation retract is a mapping of the boundary of a shape (surface) to its skeleton [16]. In the context of black holes, we have a deformation of quantum states in their neighborhood. Here we show how, starting from the 't Hooft black hole equipped with quantum strates entangled on its horizon, an algebraic topological description of the HP can be provided.

The derivation of the holographic principle is represented concisely as a fibre bundle. Briefly, a ***fibre bundle*** is a triple $(E,\ \pi, B)$, where $\pi: E \to B$ is a projection mapping from a bundle space E to a base space B [17]. Fibre bundles are on the threshold of an operational view of a complex collection of mappings that includes a projection mapping. This is the case, since it is a straightforward task to extract from a fibre bundle the steps of an algorithm (aka, precise prescription leading to implementations in different settings, such as a black hole's horizon equipped with antipodal points). A fibre bundle representation of the holographic principle L is given in Figure 1, that illustrates how the holographic manifold $L(M, X)$ can be extracted from a cross product of mappings:

$$\begin{aligned} &X(n) \otimes M\left(m\left(\varphi(X(n)), \varphi(-X(n))\right)\right) \to L(M, X) \\ &X \otimes M \to L(M, X) \\ &L(M, X), \end{aligned} \tag{1}$$

Where $X(n)$ is the Hamiltonian of a quantum signal impacting on the particle n and an antipodal gathering $M(m(\varphi(X(n)), \varphi(-X(n))))$ that includes input from black hole's antipodal evaluations of $X(n)$ from a spherical view of the black hole's horizon, i.e., antipodal values $\varphi\big(X(n)\big), \varphi\big(-X(n)\big)$ originated from a circle-shaped region of the horizon. That is, the mapping.

$$X \otimes M \to L(M, X) \tag{2}$$

Models the mapping of the accumulation $\otimes$ accruing from the interaction of the results of the mappings X and M to the holographic manifold $L\ (M, X)$, which displays a dimension lower than the black hole's 4D surface.

Concerning Figure 1, each arrow $\longmapsto$ represents a mapping. The arrows depict both ordinary mappings that carry the derivation forward and a projection mapping from $X(n)$ to the black hole's horizon, which results in a gathering of antipodal evaluations of $X(n)$, namely, $m(\varphi\big(X(n)\big), \varphi\big(-X(n)\big))$. A particular value of a holographic manifold $L\ (M, X)$ results from a synthesis of two signals: $X(n)$ and $M(m(\varphi\big(X(n)\big), \varphi\big(-X(n)\big)))$.

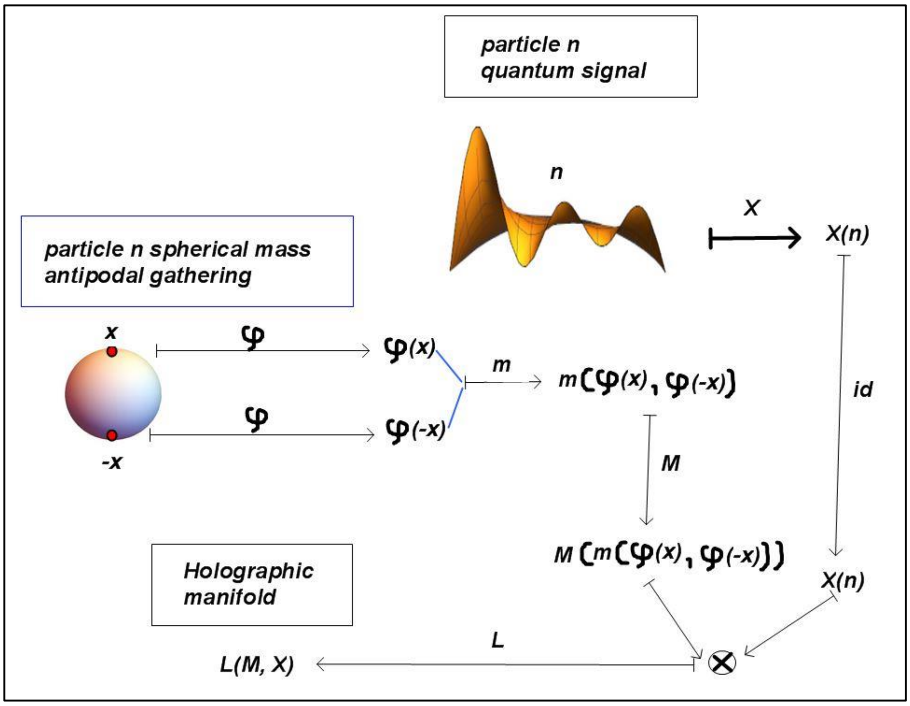


**Figure 1. Fibre bundle representation the holographic principle and its relationships with black holes. The picture illustrates the procedure to achieve a topological correlation between black hole's surface and the holographic principle. See text for further details.**

## 3- Black Hole Dynamics on a Möbius Strip

The entangled antipodal points on black hole surfaces described by ‘t Hoof trequire spacetime topology displaying a time-like Möbius strip [1]. The Möbius strip, also called the twisted cylinder, is a one-side surface equipped with just one boundary [18, 19], when embedded in three-dimensional Euclidean space. A Möbius strip can be built by taking a paper strip and giving it a half-twist, then joining the ends in order to form a loop. This means that a line that starts from the seam down the middle meets back at the seam, but at the other side. If continued, the line meets the starting point, in a point that is double the length of the original strip. This single continuous curve may be described either through a parameterized subset of a three-dimensional Euclidean space, or through cylindrical polar coordinates. Topologically, the Möbius strip can be defined as the square $[0, 1] \times [0, 1]$, with its top and bottom sides identified by the relation $(x, 0) \sim (1 - x, 1)$ for $0 \leq x \leq 1$.

In ‘t Hooft’s terms, the mapping obtained by making a trip around the black hole’s Möbius strip is a CPT inversion that allows time to change sign at the horizon.

In the previous paragraphs, we showed how the BUT, which copes with projections and mappings among different functional dimensions, has been proven suitable for the description of black hole’s antipodal features. Here we show how the BUT’s antipodal features with matching description can be assessed in terms of closed paths on a Möbius strip. This allows us to evaluate the system’s dynamics in terms of paths and trajectories taking place onto the well-established, easily manageable phase space of a twisted cylinder. Because the techniques of algebraic topology that assess the BUT features are quite complex, difficult to approach and quantify, a framework is required that allows the description of the BUT’s matching features in terms of dynamics taking place in phase spaces. This means that the scenario described by the BUT can be transported to a peculiar phase space, i.e., a Möbius strip, in order that antipodal points can be tackled in terms of trajectories taking place on a rather simple abstract manifold.

Our aim is to achieve the transport of the BUT’s antipodal points to the one-side surface of such twisted cylinder. If we embed the trajectories of two BUT matching functions x and –x (Figure 2A) on a Möbius strip, we achieve a closed, continuous loop where the two functions are allowed to travel along constrained trajectories. It is easy to see that a piece of strip of a given length, standing for a time interval, may display both x and –x at the same time (Figure 2B). The BUT dictates are preserved because, even if the two matching features are simultaneous, they do not have points in common: indeed, they lie on the opposite surface of the same strip. In black holes’ terms, this means that the oscillations’ trajectories of two areas which activate together can be followed in subsequent times, even when their matching activation has disappeared (Figure 2C).

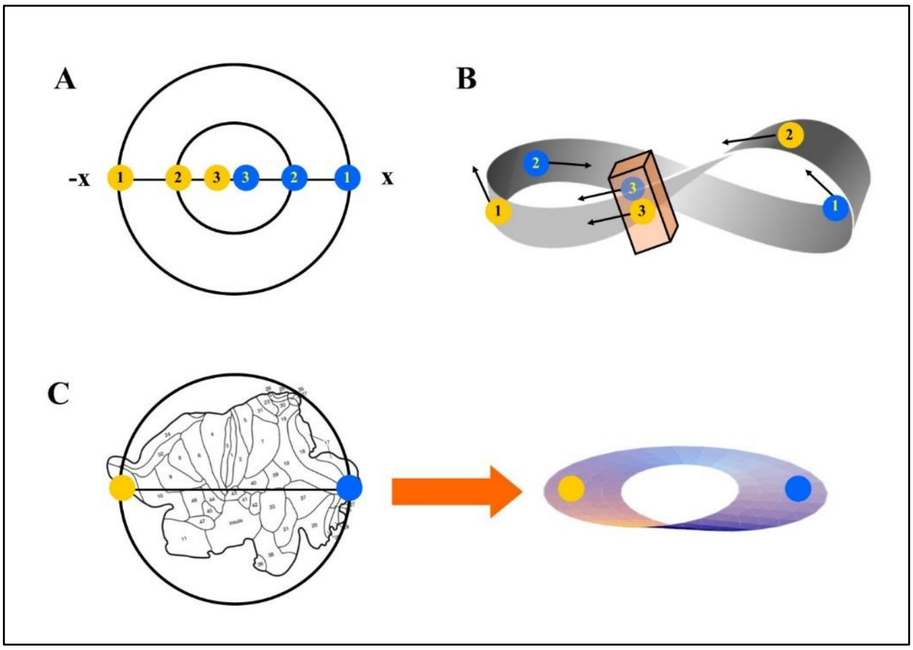


**Figure 2. Transport of the BUT theorem on a Möbius strip. Figure 2A. Changing the radius of the hypersphere makes the antipodal points more or less close. Close to the center, the two points (marked with the number 3) are almost superimposed. Figure 2B. The movements of the antipodal points can be described in terms of trajectories on a Möbius strip. The parallelepiped stands for a slice of time, where both the antipodal features occur simultaneously. Figure 2C. A theoretical example from neuroscience is provided. A cerebral hemisphere is unfolded and flattened into a two-dimensional reconstruction [26] that can be embedded into a circular manifold. When two antipodal areas display simultaneously a feature in common, e.g., the same value of pairwise entropy [27], we achieve a topological description assessable in terms of BUT (left side). Such two areas and their subsequent dynamics can be easily visualized and assessed in terms of trajectories taking place on an abstract twisted cylinder (right side).**

## 4- Introducing Time in the BUT Framework

The account of the cosmic holographic principle is generally provided by a framework which takes for granted that the event horizon is equipped with two spatial dimensions plus time. However, another possibility does exist, in order to describe a holographic manifold. A question arises: is it just a coincidence that the parameter time is not contemplated in two important formulas describing the Universe and the holographic principle? Indeed, both the Bekenstein-Hawking and Wheeler-De Witt equations take into account a static state of the related phenomena. Here the Moreva et al.'s results come into play [20]. These Authors experimentally described how an observer located inside the Universe perceives the time flow, while a hypothetical external observer perceives the Universe as motionless. According to their framework, entanglement discloses time as an emergent phenomenon. By running their experiment in two different modes ("observer" and "super-observer" mode) they showed how the same energy-entangled Hamiltonian eigenstate can be perceived as evolving by the internal observers that test the correlations between a clock subsystem and the rest, whereas it is static for the super-observer [21]. If we describe the Moreva et al.'s framework in terms of the BUT, we achieve the following topological result: an "observer" lies on a $S^3$ manifold, while a "superobserver" on a $S^2$ manifold. Indeed, the higher-dimensional manifold displays the coordinate of time, while the lower-dimensional does not. In physical terms, a manifold equipped with four dimensions (the three spatial dimensions plus time) encompasses two features with matching description. In turn, if we keep the dimension of time equal to zero (therefore removing it), we achieve a manifold, equipped with just three (spatial) dimensions, that encompasses just a single feature.

In sum, joining together the above-mentioned frameworks, it might be hypothesized that quantum entanglement occurs in spacetime, while disentanglement is achieved onto a motionless, three-dimensional manifold. Therefore, we may introduce the holographic principle in the following BUT terms: a motionless feature lying in a lower-dimensional stationary $S^2$ manifold gives rise to two moving features on a higher dimensional $S^3$ manifold, where time flow occurs. In other words, a feature encompassed in an unentangled state characterized by absence of time gives rise, when projected in one dimension higher (where time is not anymore zero), to two entangled particles. And vice versa.

## 5- Antipodal Masses: a Hypothesis

The above-mentioned frameworks allow to compare curved spacetime manifolds with structures equipped with antipodal symmetries. It is generally agreed that a black hole tends to deform the space around it, creating a vortex that captures nearby chunks of matter. The evolution of black holes can be represented by a Schwarzschild Spacetime Embedding Diagram [22]. In this approach, an embedding diagram for the vortex for a black hole can be visualized as a rubber sheet onto which a heavy mass is dropped. When an initial mass in increasing, the black hole's radius increases, burgeoning to a new mass with increasing gravitational pull. This observation allows us to tackle the issue in terms of antipodal points on black holes. Indeed, the different modes of the mass of a chunk of matter in the neighborhood of a black hole might reveal mass as an emergent phenomenon. The antipodal spacetime scenario for a chunk of matter being sucked into (of in the neighborhood of) a black hole is shown in Figure 3 in a Penrose diagram, that is a conformal compactification of 2D Minkowski space. Using such a diagram to represent the evolution of soft particles populating spacetime was first suggested by Gerard 't Hooft [1, 2].

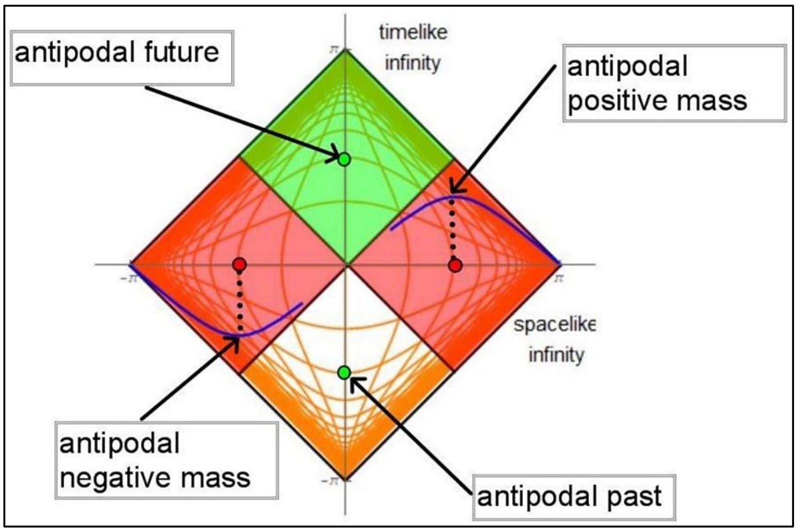


**Figure 3. Evolution of chunks of soft matter in the neighborhood of a black hole. Antipodal spacetime is represented with a Penrose diagram, which also mimics the behavior of matter in the neighborhood of a black hole. Vertically, the green region represents future time and the orange region represents past time in the lifespan of a chuck of matter. Horizontally, the concave down blue geodesic line in the red region $+\pi$ represents the positive mass of a chuck matter on the $S^4$ surface of a black hole, whereas the concave up blue geodesic line in the red region $-\pi$ represents the negative mass of a chunk of matter on the opposite surface.**

Indeed, Penrose diagrams are able to capture the causal relations between antipodal points in spacetime. It is used to represent the infinities (timelike infinities vertically in two regions representing spacetime past and future, and spacelike infinities representing the evolution of the mass of a chunk of matter on the surface of a black hole). Figure 3 extends ‘t Hooft’s model, using the horizontal axis to represent the masses of soft matter [23-25]. The relationship between chunks of matter and a black hole in the neighborhood of surrounding ones, represented by the Penrose diagram in Figure 3, says to us that, as well as it is feasible to achieve antipodal points with matching features on a black hole horizon, we are allowed to hypothesize the simultaneous presence on the horizon of particles with positive and negative mass.

## 6- Conclusion

We showed how black holes’ antipodal points can be described in terms of BUT. This led us to an algebraic topological description of the HP. The correlation between black hole’s surface and HP allows a fibre bundle representation, standing for an algorithm that can be implemented in softwares. The BUT approaches [28, 29] are fruitful, because they allow the formulation of intriguing theoretical claims, suggesting a) the possible presence of antipodal positive and negative masses on black hole horizons and b) a feature encompassed in an unentangled state characterized by absence of time might give rise to entangled particles, when mapped to a four-dimensional spacetime. Because ‘t Hooft tackles antipodal entangled quantum states in terms of opposite points on a $S^3$ hypersphere, his account might hold also for the 4-dimensional Minkowskian manifold of the general relativity. Therefore, we are in front of a potential unification of quantum mechanics and general relativity on a $S^3$ manifold.

Results from different disciplines point towards the BUT as a universal principle to quantitatively assess otherwise elusive biological/physical activities [30-32]. In this topological context, systems operations become projections among different levels, giving rise to apparently emergent properties in higher dimensions. Here we showed how the features described by BUT, occurring on an orientable manifold with positive-curvature, can be assessed in terms of paths on a non-orientable manifold, i.e., a Möbius strip. The possibility to locate oscillations on a Möbius strip allows the assessment of simultaneous activities that are spatially separated. It must also be taken into account that the BUT requirements, such as two features with no points in common and the proper mappings, are fully preserved when projected to a Möbius strip. The transport of the BUT apparatus to a Möbius strip displays also another valuable advantage: because the mapping achieved by making a trip around a twisted cylinder is an inversion, this permits the preservation of the invariance under inversions, therefore obeying to the laws of conservation of energy and information [30, 33]. In sum, the study of patterns on a twisted cylinder (instead of a three-dimensional Euclidean phase space) is justified by the BUT framework and might pave the way to the detection of unexpected relationships among different synchronous physical activities.

## 7- Conflict of Interest

The authors declare no conflict of interest.

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